Review and Preview to Chapter 11

1. (a) $V = \pi(10)^2(20) = 2000\pi \text{ cm}^3$

   (b) $V = \pi(6)^2(30) = 1080\pi \text{ cm}^3$

   (c) $V = \pi(2)^2(0.6) = 2.4\pi \text{ cm}^3$

   (d) $V = \pi(12.5)^2(25) = 3906.25\pi \text{ cm}^3$

   (e) $V = \pi\left[(7.5)^2(5) + (3)^2(4)\right] = 317.25\pi \text{ cm}^3$

   (f) $V = \pi\left[(12)^2 - (8)^2\right](12) = 960\pi \text{ cm}^3$

Exercise 11.1

1. (a) $a = 0$, $b = 4$, and $f(x) = x^2 - 2x$, so,

$\Delta x = \frac{4-0}{n} = \frac{4}{n}$ and $x_i = 0 + \frac{4i}{n} = \frac{4i}{n}$, and,

$$\int_0^4 (x^2 - 2x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2 - 2\left(\frac{4i}{n}\right)\right]\frac{4}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{64i^2}{n^3} - \frac{32i}{n^2}\right] = \lim_{n\to\infty}\left[\frac{64}{n^3}\sum_{i=1}^{n} i^2 - \frac{32}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{64}{n^3}\right)\frac{n(n+1)(2n+1)}{6} - \left(\frac{32}{n^2}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{64}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{32}{2}\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{64}{6}(1)(2) - \frac{32}{2}(1) = \frac{128}{6} - \frac{32}{2} = \frac{16}{3}$$

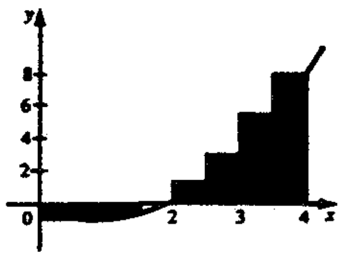
(b) $\Delta x = \frac{4-0}{8} = \frac{1}{2}$. Therefore,

$$\sum_{i=1}^{8} f(x_i)\Delta x = \frac{1}{2}\sum_{i=1}^{8} f(x_i)$$

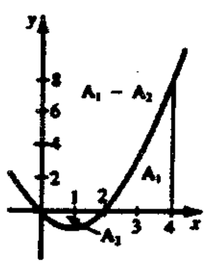
$$= \frac{1}{2}[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4)]$$

$$= \frac{1}{2}[-0.75 - 1 - 0.75 + 0 + 1.25 + 3 + 5.25 + 8] = 7.5$$

(c)



(d) Since $f(x) = x^2 - 2x \le 0$ for $0 \le x \le 2$, and $f(x) \ge 0$ for $x \ge 2$, the integral can be interpreted as $A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.



Exercise 11.1

2. (a) $a = -2$ and $b = 3$, so,

$\Delta x = \frac{3-(-2)}{n} = \frac{5}{n}$ and $x_i = -2 + \frac{5i}{n}$, and,

$$\int_{-2}^{3}(1-4x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-2+\frac{5i}{n}\right)\frac{5}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[1-4\left(-2+\frac{5i}{n}\right)\right]\frac{5}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(9-\frac{20i}{n}\right)\frac{5}{n} = \lim_{n\to\infty}\left(\frac{45}{n}\sum_{i=1}^{n} - \frac{100}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty}\left[\frac{45}{n}n - \left(\frac{100}{n^2}\right)\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[45 - 50(1)\left(1+\frac{1}{n}\right)\right]$$

$$= 45 - 50 = -5$$

(b) $a = 0$ and $b = 1$, so,

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = 0 + \frac{i}{n} = \frac{i}{n}$, and,

$$\int_{0}^{1}(1+4x-6x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[1+4\left(\frac{i}{n}\right)-6\left(\frac{i}{n}\right)^2\right]\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}+\frac{4i}{n^2}-\frac{6i^2}{n^3}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} + \frac{4}{n^2}\sum_{i=1}^{n} i - \frac{6}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}n + \left(\frac{4}{n^2}\right)\frac{n(n+1)}{2} - \left(\frac{6}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[1 + 2(1)\left(1+\frac{1}{n}\right) - 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 1 + 2 - 2 = 1$$

(c) $a = 0$ and $b = 1$, so,

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = 0 + \frac{i}{n} = \frac{i}{n}$, and,

$$\int_{0}^{1}x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^4}\right)\frac{n^2(n+1)^2}{4} = \lim_{n\to\infty}\frac{1}{4}(1)\left(1+\frac{1}{n}\right)^2 = \frac{1}{4}$$

(d) $a = 1$ and $b = 4$, so,

$\Delta x = \frac{4-1}{n} = \frac{3}{n}$ and $x_i = 1 + \frac{3i}{n}$, and,

$$\int_{1}^{4}(x^2-6)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2-6\right]\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}-6\right)\frac{3}{n} = \lim_{n\to\infty}\left(-\frac{15}{n}\sum_{i=1}^{n} + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left[-\frac{15}{n}n + \left(\frac{18}{n^2}\right)\frac{n(n+1)}{2} + \left(\frac{27}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[-15 + 9(1)\left(1+\frac{1}{n}\right) + \frac{27}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = -15 + 9 + \frac{27}{6}(2) = 3$$

3. (a) a = 0 and $b = 3$, so,

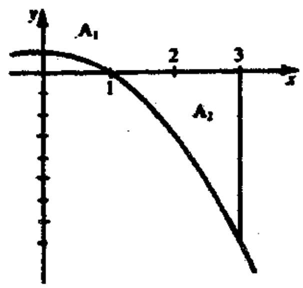
$x = \frac{3}{n}$ and $x_i = \frac{3i}{n}$, and,

$$\int_0^3 (1-x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[1-\left(\frac{3i}{n}\right)^2\right]\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}-\frac{27i^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n} 1 - \frac{27}{n^3}\sum_{i=1}^{n} i^2\right) = \lim_{n\to\infty}\left[\frac{3}{n}n - \left(\frac{27}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[3 - \frac{27}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 3 - \frac{27}{6}(2) = -6$$

Since $1 - x^2 \geq 0$ for $0 \leq x \leq 1$, and $1 - x^2 \leq 0$ for $x \geq 1$, the integral $\int_0^3 (1-x^2)\,dx$ can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in the diagram.



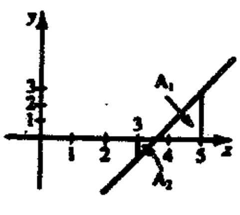
(b) a = 3 and $b = 5$, so,

$\Delta x = \frac{5-3}{n} = \frac{2}{n}$ and $x_i = 3 + \frac{2i}{n}$, and,

$$\int_3^5 (2x-7)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[2\left(3+\frac{2i}{n}\right)-7\right]\frac{2}{n} = \lim_{n\to\infty}\left(-\frac{2}{n}\sum_{i=1}^{n} + \frac{8}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty}\left[-\frac{2}{n}n + \left(\frac{8}{n^2}\right)\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[-2 + 4(1)\left(1+\frac{1}{n}\right)\right] = -2 + 4 = 2$$

Since $2x - 7 \geq 0$ for $x \geq \frac{7}{2}$, and $2x - 7 \leq 0$ for $x \leq \frac{7}{2}$, the integral $\int_3^5 (2x-7)\,dx$ can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in the diagram.



4. $\Delta x = \frac{b-a}{n}$ and $x_i = a + \left(\frac{b-a}{n}\right)i$, and,

$$\int_a^b x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left[a + \left(\frac{b-a}{n}\right)i\right]\left(\frac{b-a}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left[a + \left(\frac{b-a}{n}\right)i\right]^2\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[a^2 + 2a\left(\frac{b-a}{n}\right)i + \left[\left(\frac{b-a}{n}\right)i\right]^2\right]\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty}\left[a^2\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} + 2a\left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^3\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[a^2\left(\frac{b-a}{n}\right)n + 2a\left(\frac{b-a}{n}\right)^2\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^3\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[a^2(b-a) + a(b-a)^2(1)\left(1+\frac{1}{n}\right) + \frac{(b-a)^3}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= a^2(b-a) + a(b-a)^2 + \tfrac{1}{3}(b-a)^3 = \tfrac{1}{3}(b^3 - a^3)$$

5. (a) Using the method developed in Chapter 5 to sketch $f(x) = x^3 - 4x$, we have for headings A – H:

A. The domain is R.

B. The y-intercept is $f(0) = 0$. The x-intercepts occur when $y = 0$, so they are $0, \pm 2$.

C. $f(-x) = -x^3 + 4x = -f(x)$, therefore, $f(x) = x^3 - 4x$ is an odd function. The curve is symmetric about the origin.

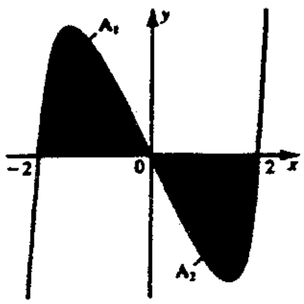
D. $\lim_{n\to\infty} x^3 - 4x = \infty$ and $\lim_{n\to-\infty} x^3 - 4x = -\infty$, so, there are no horizontal asymptotes. The denominator of $x^3 - 4x$ is 1, so there are no vertical asymptotes.

E. $f'(x) = 3x^2 - 4$, so, $f'(x) > 0$ when $x > \frac{2}{\sqrt{3}}$ or $x < -\frac{2}{\sqrt{3}}$ and $f'(x) < 0$ when $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$ Therefore, f is increasing on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ and decreasing on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

F. $f'(x) = 0$ for $x = \pm\frac{2}{\sqrt{3}}$ Therefore, by the First Derivative Test, $f\left(-\frac{2}{\sqrt{3}}\right) = \frac{16\sqrt{3}}{9}$ is a local maximum, and $f\left(\frac{2}{\sqrt{3}}\right) = -\frac{16\sqrt{3}}{9}$ is a local minimum.

G. $f''(x) = 6x \Rightarrow f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$, so f is concave upward on $(0, \infty)$, and concave downward on $(-\infty, 0)$. Thus (0,0) is the point of inflection.

H.



(b) $a = -2$, and $b = 2$, so,

$\Delta x = \frac{4}{n}$ and $x_i = -2 + \frac{4i}{n}$, and,

$$\int_{-2}^{2} (x^3 - 4x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2 + \frac{4i}{n}\right)\left(\frac{4}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-2 + \frac{4i}{n}\right)^3 - 4\left(-2 + \frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-8 + \frac{48i}{n} - \frac{96i^2}{n^2} + \frac{64i^3}{n^3} + 8 - \frac{16i}{n}\right)\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{256}{n^4}\sum_{i=1}^{n} i^3 - \frac{384}{n^3}\sum_{i=1}^{n} i^2 + \frac{128}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \left[\left(\frac{256}{n^4}\right)\frac{n^2(n+1)^2}{4} - \left(\frac{384}{n^3}\right)\frac{n(n+1)(2n+1)}{6} + \left(\frac{128}{n^2}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[64(1)\left(1 + \frac{1}{n}\right)^2 - 64(1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 64(1)\left(1 + \frac{1}{n}\right)\right] = 64 - 64(2) + 64 = 0$$

(c) $f(x) = x^3 - 4x$ is an odd function, and therefore the symmetry of the curve about the origin demands that $A_1 = A_2$. The integral can be represented by the difference of areas $A_1 - A_2 = 0$. In fact, $\int_{-a}^{a} f(x)\,dx = 0$ for any integrable odd function, for the same reasons.

Exercise 11.2

1. (a) $\int_{-6}^{7} 2\,dx = [2x]_{-6}^{7} = 2(7) - 2(-6) = 26$

(b) $\int_{-1}^{5} (6x - 7)\,dx = \left[6\frac{x^2}{1+1} - 7x\right]_{-1}^{5} = \left[3(5)^2 - 7(5)\right] - \left[3(-1)^2 - 7(-1)\right] = 30$

(c) $\int_{1}^{2} (5 + 4x - 6x^2)\,dx = \left[5x + 4\frac{x^2}{1+1} - 6\frac{x^3}{2+1}\right]_{1}^{2}$

$= (10 + 8 - 16) - (5 + 2 - 2) = -3$

(d) $\int_{0}^{1} (t^2 + 6t - 1)\,dt = \left[\frac{1}{3}t^3 + 3t^2 - t\right]_{0}^{1} = \frac{1}{3} + 3 - 1 = \frac{7}{3}$

(e) $\int_{-1}^{2} (x^3 - x^2 + 4x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + 2x^2\right]_{-1}^{2} = \left(\frac{16}{4} - \frac{8}{3} + 8\right) - \left(\frac{1}{4} + \frac{1}{3} + 2\right) = \frac{27}{4}$

(f) $\int_{0}^{1} (x^{99} + 1)\,dx = \left[\frac{1}{100}x^{100} + x\right]_{0}^{1} = \frac{1}{100} + 1 = 1.01$

(g) $\int_{2}^{3} \left(\frac{1}{t^2}\right)dt = \left[\frac{t^{-1}}{-2+1}\right]_{2}^{3} = \left[-t^{-1}\right]_{2}^{3} = -\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{6}$

(h) $\int_{1}^{4} (x - \sqrt{x})\,dx = \left[\frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3}\right]_{1}^{4} = \left(8 - \frac{16}{3}\right) - \left(\frac{1}{2} - \frac{2}{3}\right) = \frac{17}{6}$

(i) $\int_{0}^{1} \sqrt[4]{x^5}\,dx = \left[\frac{4x^{\frac{9}{4}}}{9}\right]_{0}^{1} = \frac{4}{9}$

(j) $\int_{1}^{8} \frac{2}{\sqrt[3]{x}}\,dx = \left[3x^{\frac{2}{3}}\right]_{1}^{8} = 3(4 - 1) = 9$

(k) $\int_{1}^{2} \frac{x^3 + x^2 + 1}{x^3}\,dx = \int_{1}^{2}\left(1 + \frac{1}{x} + \frac{1}{x^3}\right)dx = \left[x + \ln|x| - \frac{x^{-2}}{2}\right]_{1}^{2}$

$= \left(2 + \ln 2 - \frac{1}{8}\right) - \left(1 + \ln 1 - \frac{1}{2}\right) = \frac{11}{8} + \ln 2$

(l) $\int_{1}^{4} \left(\frac{\sqrt{x} + 1}{x}\right)dx = \int_{1}^{4}\left(\frac{1}{\sqrt{x}} + \frac{1}{x}\right) = \left[2\sqrt{x} + \ln x\right]_{1}^{4} = (4 + \ln 4) - (2 + \ln 1) = 2(1 + \ln 2)$

(m) $\int_{0}^{64} \sqrt{y}\left(1 + \sqrt[3]{y}\right)dy = \int_{0}^{64}\left(y^{\frac{1}{2}} + y^{\frac{5}{6}}\right)dy = \left[\frac{2y^{\frac{3}{2}}}{3} + \frac{6y^{\frac{11}{6}}}{11}\right]_{0}^{64} = \frac{1024}{3} + \frac{12288}{11} = \frac{48128}{33}$

(n) $\int_{0}^{\frac{\pi}{2}} (8x + \cos x)\,dx = \left[4x^2 + \sin x\right]_{0}^{\frac{\pi}{2}} = \left(\pi^2 + 1\right) - 0 = \pi^2 + 1$

(o) $\int_{0}^{\frac{\pi}{6}} (\sec x \tan x)\,dx = [\sec x]_{0}^{\frac{\pi}{6}} = \frac{2}{\sqrt{3}} - 1$

(p) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (3\sin\theta - \sec^2\theta)\,d\theta = \left[-3\cos\theta - \tan\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \left(-\frac{3}{2} - \sqrt{3}\right) - \left(-\frac{3}{\sqrt{2}} - 1\right)$

$= \frac{1}{2}(3\sqrt{2} - 1) - \sqrt{3}$

2. (a) $\int (x^5 - 2x^3 + 4)\,dx = \frac{1}{6}x^6 - \frac{1}{2}x^4 + 4x + C$

(b) $\int x^2\sqrt{x}\,dx = \int x^{\frac{5}{2}}\,dx = \frac{2x^{\frac{7}{2}}}{7} + C$

(c) $\int \left(t + \frac{2}{t}\right)dt = \frac{1}{2}t^2 + 2\ln|t| + C$

(d) $\int (1 + \sqrt{x})^2\,dx = \int (1 + 2\sqrt{x} + x)\,dx = x + \frac{4x^{\frac{3}{2}}}{3} + \frac{x^2}{2} + C$

(e) $\int \frac{x - 5}{\sqrt[4]{x}}\,dx = \int \left(x^{\frac{3}{4}} - 5x^{-\frac{1}{4}}\right)dx = \frac{4x^{\frac{7}{4}}}{7} - \frac{20x^{\frac{3}{4}}}{3} + C$

(f) $\int (\cos\theta + \sin\theta)\,d\theta = \sin\theta - \cos\theta + C$

(g) $\int (5x^4 - 2\csc x\cot x)\,dx = x^5 + 2\csc x + C$

(h) $\int (2\csc^2 x + 1)\,dx = x - 2\cot x + C$

3. (a) $\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$

(b) $\int_{-1}^{1} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{1} = \frac{1}{\ln 2}\left(2 - \frac{1}{2}\right) = \frac{3}{2\ln 2}$

(c) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^2}}\,dx = \left[\sin^{-1}x\right]_0^{\frac{1}{2}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

(d) $\int_1^{\sqrt{3}} \frac{12}{1 + x^2}\,dx = 12\left[\tan^{-1}x\right]_1^{\sqrt{3}} = 12\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \pi$

(e) $\int_{-1}^{1} \left(x + 1 + \frac{3}{1 + x^2}\right)dx = \left[\frac{x^2}{2} + x + 3\tan^{-1}x\right]_{-1}^{1}$

$= \left(\frac{1}{2} + 1 + \frac{3\pi}{4}\right) - \left(\frac{1}{2} - 1 - \frac{3\pi}{4}\right) = \frac{4 + 3\pi}{2}$

(f) $\int_{-\pi}^{0} (2e^x + \sin x)\,dx = \left[2e^x - \cos x\right]_{-\pi}^{0} = (2 - 1) - (2e^{-\pi} + 1) = -2e^{-\pi}$

4. The Fundamental Theorem of Calculus (p. 501) holds true only for functions that are continuous on the closed interval over which you wish to integrate. In this case we require $f(x) = x^{-4}$ to be continuous on $[-2, 1]$. However, f is clearly not continuous at $x = 0$ (which is in $[-2, 1]$), and therefore we cannot find $\int_{-2}^{1} x^{-4}\,dx$ using the Fundamental Theorem of Calculus as shown in the text.

Exercise 11.3

1. (a) Let $u = x^2$, then $du = 2x\,dx$.

(b) Let $u = \ln x$, then $du = \frac{dx}{x}$.

(c) Let $u = 5x$, then $du = 5\,dx$.

(d) Let $u = \sin x$, then $du = \cos x\,dx$.

2. (a) Let $u = 1 - x^2$, then $du = -2x\,dx$. So $-\frac{1}{2}du = x\,dx$, and,

$$\int x(1-x^2)^{10}dx = \int -\tfrac{1}{2}u^{10}du = -\tfrac{1}{22}u^{11} + C = -\tfrac{1}{22}(1-x^2)^{11} + C$$

(b) Let $u = 5x$, then $du = 5\,dx$. So, $\frac{1}{5}du = dx$, and,

$$\int e^{5x}dx = \int \tfrac{1}{5}e^{u}du = \tfrac{1}{5}e^{u} + C = \tfrac{1}{5}e^{5x} + C$$

(c) Let $u = x - 1$, then $du = dx$. So,

$$\int \sqrt{x-1}\,dx = \int \sqrt{u}\,du = \tfrac{2}{3}u^{\frac{3}{2}} + C = \tfrac{2}{3}(x-1)^{\frac{3}{2}} + C$$

(d) Let $u = x^2 + 2x - 6$, then $du = (2x+2)dx$. So $\frac{1}{2}du = (x+1)dx$, and,

$$\int \frac{x+1}{x^2+2x-6}dx = \int \frac{du}{2u} = \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln|x^2+2x-6| + C$$

3. (a) Let $u = x^2 + 4$, then $du = 2x\,dx$. So $\frac{1}{2}du = x\,dx$, and,

$$\int x(x^2+4)^8dx = \int \tfrac{1}{2}u^8du = \tfrac{1}{18}u^9 + C = \tfrac{1}{18}(x^2+4)^9 + C$$

(b) Let $u = x^3 + 2$, then $du = 3x^2dx$. So $\frac{1}{3}du = x^2dx$, and,

$$\int x^2\sqrt{x^3+2}\,dx = \int \tfrac{1}{3}\sqrt{u}\,du = \tfrac{2}{9}u^{\frac{3}{2}} + C = \tfrac{2}{9}(x^3+2)^{\frac{3}{2}} + C$$

(c) Let $u = x + 6$, then $du = dx$. And,

$$\int (x+6)^{10}dx = \int u^{10}du = \tfrac{1}{11}u^{11} + C = \tfrac{1}{11}(x+6)^{11} + C$$

(d) Let $u = 3x - 1$, then $du = 3\,dx$. So $\frac{1}{3}du = dx$, and,

$$\int \frac{1}{(3x-1)^2}dx = \int \frac{du}{3u^2}du = -\frac{1}{3u} + C = -\frac{1}{3(3x-1)} + C$$

(e) Let $u = 3x$, then $du = 3\,dx$. So $\frac{1}{3}du = dx$, and,

$$\int \sec^2 3x\,dx = \int \tfrac{1}{3}\sec^2 u\,du = \tfrac{1}{3}\tan u + C = \tfrac{1}{3}\tan 3x + C$$

(f) Let $u = 1 + 2x^4$, then $du = 8x^3dx$. So $\frac{1}{8}du = x^3dx$, and,

$$\int (1+2x^4)x^3dx = \int \tfrac{1}{8}u\,du = \tfrac{1}{16}u^2 + C = \tfrac{1}{16}(1+2x^4) + C$$

(g) Let $u = \sin x$, then $du = \cos x\, dx$. So,

$$\int \sin^2 x \cos x\, dx = \int u^2\, du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3 x + C$$

(h) Let $u = \ln x$, then $du = \frac{dx}{x}$. So,

$$\int \frac{\sqrt{\ln x}}{x}\, dx = \int \sqrt{u}\, du = \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2(\ln x)^{\frac{3}{2}}}{3} + C$$

(i) Let $u = t^3$, then $du = 3t^2 dt$. So $\frac{1}{3}du = t^2 dt$, and,

$$\int t^2 e^{t^3} dt = \int \tfrac{1}{3}e^u\, du = \tfrac{1}{3}e^u + C = \tfrac{1}{3}e^{t^3} + C$$

(j) Let $u = 1 - x$, then $du = -dx$. So $-du = dx$, and,

$$\int \frac{1}{1-x}\, dx = \int -\frac{du}{u} = -\ln|x| + C = -\ln|1-x| + C$$

(k) Let $u = x^3 - 2x + 1$, then $du = (3x^2 - 2)dx$. And,

$$\int \frac{3x^2 - 2}{(x^3 - 2x + 1)^3}\, dx = \int \frac{du}{u^3} du = -\frac{1}{2u^2} + C = -\frac{1}{2(x^3 - 2x + 1)^2} + C$$

(l) Let $u = \sqrt{x}$, then $du = \frac{dx}{2\sqrt{x}}$. So $2du = \frac{dx}{\sqrt{x}}$, and,

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = \int 2\sin u\, du = -2\cos u + C = -2\cos\sqrt{x} + C$$

(m) Let $u = 3 - x$, then $du = -dx$. So $-du = dx$, and,

$$\int e^{3-x}\, dx = \int -e^u\, du = -e^u + C = -e^{3-x} + C$$

(n) Let $u = \cos x$, then $du = -\sin x\, dx$. So $-du = \sin x\, dx$, and,

$$\int e^{\cos x}\sin x\, dx = \int -e^u\, du = -e^u + C = -e^{\cos x} + c$$

(o) Let $u = 1 + \tan x$, then $du = \sec^2 x\, dx$. So,

$$\int \sqrt{1 + \tan x}\, \sec^2 x\, dx = \int \sqrt{u}\, du = \tfrac{2}{3}u^{\frac{3}{2}} + C = \tfrac{2}{3}(1 + \tan x)^{\frac{3}{2}} + C$$

(p) Let $u = x^2$, then $du = 2x\, dx$. So $\frac{1}{2}du = x\, dx$, and,

$$\int x\sin(x^2)\, dx = \int \tfrac{1}{2}\sin u\, du = -\tfrac{1}{2}\cos u + C = -\tfrac{1}{2}\cos(x^2) + C$$

(q) Let $u = \cos x$, then $du = -\sin x\, dx$. So $-du = \sin x\, dx$, and,

$$\int \sin x \sin(\cos x)\, dx = \int -\sin u\, du = \cos u + C = \cos(\cos x) + C$$

(r) Let $u = \tan^{-1} x$, then $du = \frac{dx}{1 + x^2}$. So,

$$\int \frac{\tan^{-1} x}{1 + x^2}\, dx = \int u\, du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}(\tan^{-1} x)^2 + C$$

4. (a) Let $u = 2x + 1$, then $du = 2\,dx$.

When $x = 0$, $u = 1$. When $x = 1$, $u = 3$. So,

$$\int_0^1 e^{2x+1}\,dx = \int_1^3 \tfrac{1}{2}e^u\,du = \left[\tfrac{1}{2}e^u\right]_1^3 = \tfrac{1}{2}(e^3 - e)$$

(b) Let $u = 1 + 5x$, then $du = 5\,dx$.

When $x = 0$, $u = 1$. When $x = 2$, $u = 11$. So,

$$\int_0^2 \frac{1}{(1+5x)^4}\,dx = \int_1^{11} \frac{du}{5u^4} = \left[-\frac{1}{15u^3}\right]_1^{11} = -\tfrac{1}{15}\left(\tfrac{1}{1331} - 1\right) = \tfrac{266}{3993}$$

(c) Let $u = 4 - x^2$, then $du = -2x\,dx$.

When $x = 0$, $u = 4$. When $x = 2$, $u = 0$. So,

$$\int_0^2 x\sqrt{4 - x^2}\,dx = \int_4^0 -\tfrac{1}{2}\sqrt{u}\,du = \left[-\tfrac{1}{3}u^{\frac{3}{2}}\right]_4^0 = -\tfrac{1}{3}(0 - 8) = \tfrac{8}{3}$$

(d) Let $u = \pi t$, then $du = \pi\,dt$.

When $t = 0$, $u = 0$. When $t = 1$, $u = \pi$. So,

$$\int_0^1 \sin \pi t\,dt = \int_0^{\pi} \frac{\sin u}{\pi}\,du = \left[-\frac{\cos u}{\pi}\right]_0^{\pi} = -\tfrac{1}{\pi}(-1 - 1) = \tfrac{2}{\pi}$$

(e) Let $u = \sin\theta$, then $du = \cos\theta\,d\theta$.

When $\theta = \frac{\pi}{6}$, $u = \frac{1}{2}$. When $\theta = \frac{\pi}{2}$, $u = 1$. So,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^3\theta}\,d\theta = \int_{\frac{1}{2}}^{1} \frac{du}{u^3} = \left[-\frac{1}{2u^2}\right]_{\frac{1}{2}}^{1} = -\tfrac{1}{2}(1 - 4) = \tfrac{3}{2}$$

(f) Let $u = x^5 + 1$, then $du = 5x^4\,dx$.

When $x = 0$, $u = 1$. When $x = 1$, $u = 2$. So,

$$\int_0^1 x^4(x^5 + 1)^5\,dx = \int_1^2 \tfrac{1}{5}u^5\,du = \left[\tfrac{1}{30}u^6\right]_1^2 = \tfrac{1}{30}(64 - 1) = \tfrac{21}{10} = 2.1$$

(g) Let $u = 1 + \frac{1}{x}$, then $du = -\frac{dx}{x^2}dx$.

When $x = \frac{1}{2}$, $u = 3$. When $x = 1$, $u = 2$. So,

$$\int_{\frac{1}{2}}^{1} \frac{\left(1 + \frac{1}{x}\right)^5}{x^2}\,dx = \int_3^2 -u^5\,du = \left[-\tfrac{1}{6}u^6\right]_3^2 = -\tfrac{1}{6}(64 - 729) = \tfrac{665}{6}$$

(h) Let $u = 3x^2 + 6x - 4$, then $du = 6(x + 1)\,dx$.

When $x = 1$, $u = 5$. When $x = 2$, $u = 20$. So,

$$\int_1^2 (x + 1)e^{3x^2+6x-4}\,dx = \int_5^{20} \tfrac{1}{6}e^u\,du = \left[\tfrac{1}{6}e^u\right]_5^{20} = \tfrac{1}{6}(e^{20} - e^5)$$

5. (a) Let $u = \cos x$, then $du = -\sin x\,dx$. So,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C$$

(b) Let $u = \sin x$, then $du = \cos x\,dx$. So,

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

6. Since $f(x) = \sqrt{4x+1} \geq 0$ on $[0, 10]$, then $\int_0^{10} \sqrt{4x+1}\,dx$ can be interpreted as the area between the curve of f and the x-axis.

Let $u = 4x + 1$, then $du = 4\,dx$.

When $x = 0$, $u = 1$. When $x = 10$, $u = 41$. So,

$$A = \int_0^{10} \sqrt{4x+1}\,dx = \int_1^{41} \tfrac{1}{4}\sqrt{u}\,du = \left[\tfrac{1}{6}u^{\frac{3}{2}}\right]_1^{41} = \tfrac{1}{6}\left(41^{\frac{3}{2}} - 1\right)$$

7. Since $f(x) = \cos\left(\frac{x}{2}\right) \geq 0$ on $[0, \pi]$, then $\int_0^{\pi} \cos\left(\frac{x}{2}\right)dx$ can be interpreted as the area between the curve of f and the x-axis.

Let $u = \frac{x}{2}$, then $du = \frac{1}{2}dx$.

When $x = 0$, $u = 0$. When $x = \pi$, $u = \frac{\pi}{2}$. So,

$$A = \int_0^{\pi} \cos\left(\tfrac{x}{2}\right)dx = \int_0^{\frac{\pi}{2}} 2\cos u\,du = [2\sin u]_0^{\frac{\pi}{2}} = 2(1 - 0) = 2$$

8. Since $y = e^{2x} \geq y = e^{-x}$ on $[0, 1]$, the area bounded by these curves on the given interval is:

$$A = \int_0^1 \left(e^{2x} - e^{-x}\right)dx = \int_0^1 e^{2x}\,dx - \int_0^1 e^{-x}\,dx$$

For $\int_0^1 e^{2x}\,dx$, Let $u = 2x$, then $du = 2\,dx$.

When $x = 0$, $u = 0$. When $x = 1$, $u = 2$. So,

$$\int_0^1 e^{2x}\,dx = \int_0^2 \tfrac{1}{2}e^u\,du = \left[\tfrac{1}{2}e^u\right]_0^2 = \tfrac{1}{2}(e^2 - 1)$$

For $\int_0^1 e^{-x}\,dx$, let $u = -x$, then $du = -dx$.

When $x = 0$, $u = 0$. When $x = 1$, $u = -1$. So,

$$\int_0^1 e^{-x}\,dx = \int_0^{-1} -e^u\,du = [-e^u]_0^{-1} = 1 - e^{-1}$$

Therefore, $A = \frac{1}{2}(e^2 - 1) - (1 - e^{-1}) = \frac{1}{2}(e^2 - 3) + e^{-1}$

9. (a) Let $u = \sqrt{x} + 1$, then $du = \frac{dx}{2\sqrt{x}}$. So,

$$\int \frac{1}{x + \sqrt{x}}dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx = \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(\sqrt{x}+1) + C$$

[OR: Let $u = \sqrt{x}$, then $u^2 = x$, so $dx = 2u\,du$ and

$$\int \frac{1}{x+\sqrt{x}}dx = \int \frac{2u\,du}{u^2+u} = 2\int \frac{1}{u+1}du = 2\ln|u+1| + C = 2\ln(\sqrt{x}+1) + C$$]

(b) Let $u = x + 2$, then $x + 1 = u - 1$, and $du = dx$. So,

$$\int \frac{x+1}{x+2}dx = \int \frac{u-1}{u}du = \int\left(1 - \tfrac{1}{u}\right)du = u - \ln|u| + C = x + 2 - \ln|x+2| + C$$

Exercise 11.4

1. (a) $u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

(b) $u = x \qquad dv = e^{2x}\,dx$

$du = dx \qquad v = \frac{1}{2}e^{2x}$

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

(c) $u = \ln x \qquad dv = x\,dx$

$du = \frac{dx}{x} \qquad v = \frac{1}{2}x^2$

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

(d) $u = t \qquad dv = \sec^2 t\,dt$

$du = dt \qquad v = \tan t$

$$\int t\sec^2 t\,dt = t\tan t - \int \tan t\,dt = t\tan t - \ln|\sec t| + C$$ (see 11.3, Ex. 5. (a))

(e) $u = x^2 \qquad dv = e^x\,dx$

$du = 2x\,dx \qquad v = e^x$

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x dx = x^2e^x - 2\int xe^x\,dx = (x^2-2x+2)e^x + C$$ (see E.g. 1 of 11.4)

(f) $u = 3x - 5 \qquad dv = e^{-4x}\,dx$

$du = 3\,dx \qquad v = -\frac{1}{4}e^{-4x}$

$$\int(3x-5)e^{-4x}\,dx = -\frac{1}{4}(3x-5)e^{-4x} + \int\frac{3}{4}e^{-4x}\,dx = \frac{3}{4}(5-3x) - \frac{3}{16}e^{-4x} + C$$
$$= \left(\frac{17}{16} - \frac{3}{4}x\right)e^{-4x} + C$$

(g) $u = \tan^{-1}x \qquad dv = dx$

$du = \frac{dx}{x^2+1} \qquad v = x$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x}{x^2+1}\,dx$$

Let $u = x^2 + 1$, then $du = 2x\,dx$. So,

$$\int\frac{x}{x^2+1}\,dx = \int\frac{du}{2u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$$

Therefore, $\int\tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + C$

(h) $u = xe^x \qquad dv = \frac{dx}{(x+1)^2}$

$du = (x+1)e^x\,dx \qquad v = -\frac{1}{x+1}$

$$\int\frac{xe^x}{(x+1)^2}\,dx = -\frac{xe^x}{x+1} + \int\frac{(x+1)e^x}{(x+1)}\,dx = -\frac{xe^x}{x+1} + e^x + C = \frac{e^x}{x+1} + C$$

2. (a) $u = x \qquad dv = \sin x\,dx$

$du = dx \qquad v = -\cos x$

$$\int_0^\pi x\sin x\,dx = [-x\cos x]_0^\pi + \int_0^\pi \cos x\,dx = \pi + [\sin x]_0^\pi = \pi$$

(b) $u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^1 xe^{-x}\,dx = [-xe^{-x}]_0^1 + \int_0^1 e^{-x}\,dx = -\frac{1}{e} - [e^{-x}]_0^1 = 1 - \frac{2}{e}$$

(c) $u = \ln x \qquad dv = x^4\,dx$

$du = \frac{dx}{x} \qquad v = \frac{1}{5}x^5$

$$\int_1^2 x^4 \ln x\,dx = \left[\frac{1}{5}x^5 \ln x\right]_1^2 - \int_1^2 \frac{1}{5}x^4\,dx = \frac{32}{5}\ln 2 - \left[\frac{1}{25}x^5\right]_1^2 = \frac{32}{5}\ln 2 - \frac{31}{25}$$

(d) $u = x^2 \qquad dv = \cos x\,dx$

$du = 2x\,dx \qquad v = \sin x\,dx$

$$\int_0^{2\pi} x^2 \cos x\,dx = [x^2 \sin x]_0^{2\pi} - \int_0^{2\pi} 2x \sin x\,dx = 0 - \int_0^{2\pi} 2x\sin x\,dx = -2\int_0^{2\pi} x \sin x\,dx$$

To solve $\int_0^{2\pi} x \sin x\,dx$, we integrate by parts again using the same substitutions for u and v as those used in part (a).

$u = x \qquad dv = \sin x\,dx$

$du = dx \qquad v = -\cos x$

$$\int_0^{2\pi} x \sin x\,dx = [-x\cos x]_0^{2\pi} + \int_0^{2\pi} \cos x\,dx = -2\pi + [\sin x]_0^{2\pi} = -2\pi$$

Therefore, $\int_0^{2\pi} x^2 \sin x\,dx = -2(-2\pi) = 4\pi$

3. (a) for $n \geq 1, u = x^n \qquad dv = e^x\,dx$

$du = nx^{n-1}dx \qquad v = e^x$

$$\int_0^1 x^n e^x\,dx = [x^n e^x]_0^1 - \int_0^1 nx^{n-1}dx = e - n\int_0^1 x^{n-1}dx$$

(b) Using the above formula, we have;

$$\int_0^1 x^3 e^x\,dx = e - 3\int_0^1 x^2 e^x\,dx = e - 3\left[e - 2\int_0^1 xe^x\,dx\right] = -2e + 6\int_0^1 xe^x\,dx$$

$$= -2e + 6\left[e - \int_0^1 e^x\,dx\right] = 4e - 6[e]_0^1 = 6 - 2e$$

4. Let $t = \sqrt{x}$, then $dt = \frac{dx}{2\sqrt{x}}$, or $2t\,dt = dx$. So,

$$\int e^{\sqrt{x}}\,dx = \int 2te^t\,dt$$

$u = 2t \qquad dv = e^t\,dt$

$du = 2dt \qquad v = e^t$

$$\int e^{\sqrt{x}}\,dx = \int 2te^t\,dt = 2te^t - \int 2e^t\,dt = 2te^t - 2e^t + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$

5. Since $xe^{-3x} \geq 0$ on $[0, 2]$, $A = \int_0^2 xe^{-3x}\,dx$.

$$u = x \qquad dv = e^{-3x}\,dx$$
$$du = dx \qquad v = -\tfrac{1}{3}e^{-3x}$$

$$A = \int_0^2 xe^{-3x}\,dx = \left[-\tfrac{1}{3}xe^{-3x}\right]_0^2 + \int_0^2 \tfrac{1}{3}e^{-3x}\,dx = -\tfrac{2}{3}e^{-6} - \left[\tfrac{1}{9}e^{-3x}\right]_0^2$$
$$= -\tfrac{2}{3}e^{-6} - \tfrac{1}{9}(e^{-6} - 1) = \tfrac{1}{9}(1 - 7e^{-6})$$

6. Since the x-intercept of $y = \ln x$ is 1, $A = \int_1^5 \ln x\,dx$.

$$u = \ln x \qquad dv = dx$$
$$du = \frac{dx}{x} \qquad v = x$$

$$A = \int_1^5 \ln x\,dx = [x\ln x]_1^5 - \int_1^5 dx = 5\ln 5 - \ln 1 - [x]_1^5 = 5\ln 5 - 4$$

7.
$$u = \sin x \qquad dv = e^x\,dx$$
$$du = \cos x\,dx \qquad v = e^x$$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

Now, take $\int e^x \cos x\,dx$ and integrate it by parts;

$$u = \cos x \qquad dv = e^x\,dx$$
$$du = -\sin x\,dx \qquad v = e^x$$

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$$

Therefore,

$$\int e^x \sin x\,dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x\,dx\right]$$
$$\int e^x \sin x\,dx = e^x(\sin x - \cos x) - \int e^x \sin x\,dx$$
$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$
$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Exercise 11.5

1. (a) Let $u = \sin x$, then $du = \cos x\,dx$. So,
$$\int \cos^3 x\,dx = \int (1 - \sin^2 x)\cos x\,dx = \int (1 - u^2)\,du = u - \tfrac{1}{3}u^3 + C$$
$$= \sin x - \tfrac{1}{3}\sin^3 x + C$$

(b) Let $u = \cos x$, then $du = -\sin x\,dx$. So,
$$\int \sin^3 x\cos^2 x\,dx = \int \sin x(\sin^2 x)\cos^2 x\,dx = \int (1 - \cos^2 x)\cos^2 x\sin x\,dx$$
$$= -\int (1 - u^2)u^2\,du = -\tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + C = -\tfrac{1}{3}\cos^3 x + \tfrac{1}{5}\cos^5 x + C$$

(c) $$\int \sin^2 x\cos^2 x\,dx = \int \tfrac{1}{2}(1 - \cos 2x)\tfrac{1}{2}(1 + \cos 2x)\,dx = \int \tfrac{1}{4}(1 - \cos^2 2x)\,dx$$
$$= \int \tfrac{1}{4}\left[1 - \tfrac{1}{2}(1 + \cos 4x)\right]dx = \tfrac{1}{8}x - \tfrac{1}{8}\int \cos 4x\,dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$$

(d) $$\int_0^{\frac{\pi}{2}} \sin^5 x\,dx = \int_0^{\frac{\pi}{2}} \sin^4 x\sin x\,dx = \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)^2\sin x\,dx$$
Let $u = \cos x$, then $du = -\sin x\,dx$.
When $x = 0$, $u = 1$. When $x = \frac{\pi}{2}$, $u = 0$. So,
$$\int_0^{\frac{\pi}{2}} \sin^5 x\,dx = -\int_1^0 (1 - u^2)^2\,du = \int_1^0 (-1 + 2u^2 - u^4)\,du = \left[-u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5\right]_1^0$$
$$= 0 - (-1 + \tfrac{2}{3} - \tfrac{1}{5}) = \tfrac{8}{15}$$

(e) $$\int_0^{\frac{\pi}{2}} \cos^5 x\sin^4 x\,dx = \int_0^{\frac{\pi}{2}} \cos x(\cos^4 x)\sin^4 x\,dx = \int_0^{\frac{\pi}{2}} \cos x(1 - \sin^2 x)^2\sin^4 x\,dx$$
Let $u = \sin x$, then $du = \cos x\,dx$.
When $x = 0$, $u = 0$. When $x = \frac{\pi}{2}$, $u = 1$. So,
$$\int_0^{\frac{\pi}{2}} \cos^5 x\sin^4 x\,dx = \int_0^1 (1 - u^2)^2u^4\,du = \int_0^1 (u^4 - 2u^6 + u^8)\,du = \left[\tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9\right]_0^1$$
$$= \tfrac{1}{5} - \tfrac{2}{7} + \tfrac{1}{9} = \tfrac{8}{315}$$

(f) $$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x\sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \tfrac{1}{2}(1 - \cos 2x)\tfrac{1}{2}(1 - \cos 2x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{4}(1 - 2\cos 2x + \cos^2 2x)\,dx = \int_0^{\frac{\pi}{2}} \tfrac{1}{4}\left[1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)\right]dx$$
$$= \int_0^{\frac{\pi}{2}} \left[\tfrac{3}{8} - \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x\right]dx = \left[\tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x\right]_0^{\frac{\pi}{2}} = \tfrac{3}{16}\pi$$

2. (a) Let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$. So,

$$\int x^3\sqrt{1-x^2}\,dx = \int \sin^3\theta\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int \sin^3\theta\cos^2\theta\,d\theta$$

$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C$ (see Ex. 1. (b))

Since $\sin\theta = x$, then $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. So,

$$\int x^3\sqrt{1-x^2}\,dx = \tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta + C = \tfrac{1}{5}(1-x^2)^{\frac{5}{2}} - \tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$

(b) Let $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$. When $x = 2$, $\theta = \frac{\pi}{2}$. So,

$$\int_0^2 x^2\sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{2}} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} 16\sin^2\theta\cos^2\theta\,d\theta$$

$= 16\left[\frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right)\right]_0^{\frac{\pi}{2}} = 16\left(\frac{1}{8}\right)\left(\frac{\pi}{2}\right) = \pi$ (see Ex. 1. (c))

3. (a) Let $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$.

When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{4}$. So,

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan^3\theta\sec^2\theta}{\sqrt{\tan^2\theta+1}}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^3\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta = \int_0^{\frac{\pi}{4}} \tan^3\theta\sec\theta\,d\theta$$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3\theta\sec\theta\,d\theta = \int_0^{\frac{\pi}{4}} \tan\theta(\tan^2\theta)\sec\theta\,d\theta = \int_0^{\frac{\pi}{4}} \tan\theta(\sec^2\theta - 1)\sec\theta\,d\theta$

Let $u = \sec\theta$, then $du = \sec\theta\tan\theta\,d\theta$.

When $\theta = 0$, $u = 1$. When $\theta = \frac{\pi}{4}$, $u = \sqrt{2}$. So,

$$\int_0^{\frac{\pi}{4}} \tan^3\theta\sec\theta\,d\theta = \int_1^{\sqrt{2}} (u^2-1)\,du = \left[\tfrac{1}{3}u^3 - u\right]_1^{\sqrt{2}} = \left(\tfrac{2\sqrt{2}}{3} - \sqrt{2}\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{1}{3}\left(2 - \sqrt{2}\right)$$

4. The ellipse is symmetric with respect to both axes, so the total area A is four times the area in the first quadrant. The part of the ellipse in the first quadrant is given by

$y = \frac{4}{3}\sqrt{9-x^2}$, $0 \le x \le 3$.

Therefore, $A = 4\displaystyle\int_0^{\frac{\pi}{2}} \frac{4}{3}\sqrt{9-x^2}\,dx$

Let $x = 3\sin\theta$, then $dx = 3\cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \frac{\pi}{2}$. So,

$$A = \frac{16}{3}\int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta = 48\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 24\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$= 24\left[1 + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = 12\pi$

5. (a) Let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$. So,

$$\int x\sqrt{1-x^2}\,dx = \int \sin\theta\sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \int \sin\theta\cos^2\theta\, d\theta$$

Next, we make another substitution in order to solve this integral.

Let $u = \cos\theta$, then $du = -\sin\theta\, d\theta$. So,

$$\int \sin\theta\cos^2\theta\, d\theta = -\int u^2\,du = -\tfrac{1}{3}u^3 + C = -\tfrac{1}{3}\cos^3\theta + C$$

Since $x = \sin\theta$, then $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. So,

$$\int x\sqrt{1-x^2}\,dx = -\tfrac{1}{3}\cos^3\theta + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$

(b) Let $u = 1 - x^2$, then $du = -2x\,dx$. So,

$$\int x\sqrt{1-x^2}\,dx = -\int \tfrac{1}{2}\sqrt{u}\,du = -\frac{1}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$

6. The area of an ellipse in standard position is four times the area enclosed by the ellipse in the first quadrant. The equation of this part of the ellipse is

$y = \frac{b}{a}\sqrt{a^2 - x^2}$, $0 \le x \le a$.

Therefore, $A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

Let $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$.

When $x = 0$, $\theta = 0$. When $x = a$, $\theta = \frac{\pi}{2}$. So,

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = 4\int_0^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2 - a^2\sin^2\theta}\,a\cos\theta\, d\theta = 4\int_0^{\frac{\pi}{2}} ab\cos^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} ab(1+\cos 2\theta)\,d\theta = 2ab\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\frac{\pi}{2}} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

EXERCISE 11.6

1. $\dfrac{1}{(x+2)(x-3)}=\dfrac{A}{x+2}+\dfrac{B}{x-3}$

2. $\dfrac{x+3}{(x+2)(x+5)^2}=\dfrac{A}{x+2}+\dfrac{B}{x+5}+\dfrac{C}{(x+5)^2}$

3. $\dfrac{x^2+x+1}{(x-1)(x+1)^2(x-2)^3}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}+\dfrac{D}{x-2}+\dfrac{E}{(x-2)^2}+\dfrac{F}{(x-2)^3}$

4. $\dfrac{5x}{(x^2+x+1)(x-7)}=\dfrac{Ax+B}{x^2+x+1}+\dfrac{C}{x-7}$

5. $\dfrac{2-3x}{(x+5)(x^2+4)(x^2+2x+6)}=\dfrac{A}{x+5}+\dfrac{Bx+C}{x^2+4}+\dfrac{Dx+E}{x^2+2x+6}$

6. $\dfrac{2x-1}{x^2-16}=\dfrac{2x-1}{(x-4)(x+4)}=\dfrac{A}{x-4}+\dfrac{B}{x+4}$

7. $\dfrac{x^2+1}{x^2+7x+12}=1-\dfrac{7x+11}{(x+3)(x+4)}=1-\left(\dfrac{A}{x+3}+\dfrac{B}{x+4}\right)$

8. $\dfrac{x^3-2x^2+2}{(x-5)^3(x^2+5x+10)^2}$

$$=\frac{A}{x-5}+\frac{B}{(x-5)^2}+\frac{C}{(x-5)^3}+\frac{Dx+E}{x^2+5x+10}+\frac{Fx+G}{(x^2+5x+10)^2}$$

9. Let $\dfrac{1}{x^2-1}=\dfrac{A}{x-1}+\dfrac{B}{x+1}\Rightarrow A(x+1)+B(x-1)=1\Rightarrow (A+B)x+A-B=1$

$\Rightarrow A+B=0$ and $A-B=1\Rightarrow A=\frac{1}{2}$ and $B=-\frac{1}{2}$.

Therefore $\displaystyle\int\frac{dx}{x^2-1}=\int\frac{dx}{2(x-1)}-\int\frac{dx}{2(x+1)}=\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C$

$=\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right|+C=\ln\sqrt{\dfrac{x-1}{x+1}}+C$

10. Since $\frac{2t+3}{t-1} = 2 + \frac{5}{t-1}$,

therefore $\int_2^5 \frac{2t+3}{t-1}dt = 2\int_2^5 dt + 5\int_2^5 \frac{dt}{t-1} = \left[2t + 5\ln|t-1|\right]_2^5$

$= 10 + 5\ln 4 - 4 - 5\ln 1 = 6 + 5\ln 4$

11. $\frac{x^3 - 3x^2 + x}{x^2 - 3x + 2} = x - \frac{x}{x^2 - 3x + 2}$. Let $\frac{x}{x^2 - 3x + 2} = \frac{A}{x-2} + \frac{B}{x-1}$

$\Rightarrow A(x-1) + B(x-2) = x \Rightarrow (A+B)x - A - 2B = x$

$\Rightarrow A + B = 1$ and $-A - 2B = 0 \Rightarrow A = 2$ and $B = -1$.

Therefore $\int \frac{x^3 - 3x^2 + x}{x^2 - 3x + 2}dx = \int x\,dx - \int \frac{2}{x-2}dx + \int \frac{dx}{x-1}$

$= \frac{1}{2}x^2 - 2\ln|x-2| + \ln|x-1| + C$

12. $\frac{t^3}{t^2 + 7t + 12} = t + \frac{37t + 84}{t^2 + 7t + 12}$.

Let $\frac{37t + 84}{t^2 + 7t + 12} = \frac{A}{t+3} + \frac{B}{t+4}$

$\Rightarrow 37t + 84 = A(t+4) + B(t+3) \Rightarrow (A+B)t + 4A + 3B = 37t + 84$

$\Rightarrow A + B = 37$ and $4A + 3B = 84 \Rightarrow A = -27$ and $B = 64$.

Therefore $\int \frac{t^3}{t^2 + 7t + 12}dt = \int t\,dt - 27\int \frac{dt}{t+3} + 64\int \frac{dt}{t+4}$

$= \frac{1}{2}t^2 - 27\ln|t+3| + 64\ln|t+4| + C$

13. Let $\dfrac{x+4}{2x-x^2-x^3}=\dfrac{A}{x}+\dfrac{B}{2+x}+\dfrac{C}{1-x}$

$\Rightarrow (2-x-x^2)A+(x-x^2)B+(2x+x^2)C=x+4$

$\Rightarrow (-A-B+C)x^2+(-A+B+2C)x+2A=x+4$

$\Rightarrow -A-B+C=0,\ -A+B+2C=1$ and $2A=4$

$\Rightarrow A=2,\ B=-\frac{1}{3}$, and $C=\frac{5}{3}$.

Therefore $\displaystyle\int\frac{x^2}{x^2+7x+12}dx=2\int\frac{dx}{x}-\frac{1}{3}\int\frac{dx}{2+x}+\frac{5}{3}\int\frac{dx}{1-x}$

$=2\ln x-\frac{1}{3}\ln|2+x|-\frac{5}{3}\ln|1-x|+K$

14. Let $\dfrac{1}{(x-1)^2(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}$

$\Rightarrow 1=(x^2-1)A+(x+1)B+(x^2-2x+1)C$

$\Rightarrow 1=(A+C)x^2+(B-2C)x+B+C-A$

$\Rightarrow A+C=0$ and $B-2C=0$ and $B+C-A=1 \Rightarrow A=-\frac{1}{4},\ B=\frac{1}{2}$, and $C=\frac{1}{4}$.

Therefore $\displaystyle\int\frac{dx}{(x-1)^2(x+1)}=-\frac{1}{4}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{(x-1)^2}+\frac{1}{4}\int\frac{dx}{x+1}$

$=-\frac{1}{4}\ln|x-1|-\dfrac{1}{2(x-1)}+\frac{1}{4}\ln|x+1|+K$

15. $\dfrac{x^3-1}{x^3+3x^2}=1-\dfrac{3x^2+1}{x^2(x+3)}$. Let $\dfrac{3x^2+1}{x^2(x+3)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+3}$

$\Rightarrow 3x^2+1=(x^2+3x)A+(x+3)B+Cx^2$

$\Rightarrow 3x^2+1=(A+C)x^2+(3A+B)x+3B$

$\Rightarrow A+C=3$ and $3A+B=0$ and $3B=1 \Rightarrow A=-\frac{1}{9},\ B=\frac{1}{3}$ and $C=\frac{28}{9}$.

Therfore $\displaystyle\int\frac{x^3-1}{x^3+3x^2}dx=\int dx-\frac{1}{9}\int\frac{dx}{x}+\frac{1}{3}\int\frac{dx}{x^2}+\frac{28}{9}\int\frac{dx}{x+3}$

$=x-\frac{1}{9}\ln|x|+\frac{1}{3x}+\frac{28}{9}\ln|x+3|+K$

16. Let $\frac{4}{x^3+4x}=\frac{A}{x}+\frac{Bx+C}{x^2+4} \Rightarrow A(x^2+4)+x(Bx+C)=4$

$\Rightarrow (A+B)x^2+Cx+4A=4$

$\Rightarrow A+B=0$ and $C=0$ and $4A=4 \Rightarrow A=1, B=-1,$ and $C=0.$

Therefore $\int_1^4 \frac{4\,dx}{x^3+4x}=\int_1^4 \frac{dx}{x}dx-\int_1^4 \frac{x}{x^2+4}dx=\left[\ln|x|-\frac{1}{2}\ln|x^2+4|\right]_1^4$

$=\ln 4-\frac{1}{2}\ln 20-\ln 1+\frac{1}{2}\ln 5=\ln\frac{4\times\sqrt{5}}{2\sqrt{5}}=\ln 2$

17. Let $\frac{1-3x}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} \Rightarrow A(x^2+x+1)+(Bx+C)(x-1)=1-3x$

$\Rightarrow (A+B)x^2+(A-B+C)x+A-C=1-3x$

$\Rightarrow A+B=0$ and $A-B+C=-3$ and $A-C=1$

$\Rightarrow A=-\frac{2}{3}, B=\frac{2}{3},$ and $C=-\frac{5}{3}.$

Therefore, $\int \frac{1-3x}{x^3-1}dx=\int \frac{-2}{3(x-1)}dx+\int \frac{2x-5}{3(x^2+x+1)}dx$

$=-\frac{2}{3}\int \frac{1}{x-1}dx+\frac{1}{3}\int \frac{2x+1}{x^2+x+1}dx+\frac{1}{3}\int \frac{-6}{(x+\frac{1}{2})^2+\frac{3}{4}}dx$

$=-\frac{2}{3}\ln|x-1|+\frac{1}{3}\ln|x^2+x+1|-2\int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}+C_1$

$=-\frac{2}{3}\ln|x-1|+\frac{1}{3}\ln|x^2+x+1|-2\left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+K$

$=-\frac{2}{3}\ln|x-1|+\frac{1}{3}\ln|x^2+x+1|-\frac{4}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+K$

18. $$\frac{4x^2+5x+4}{(x^2+1)(x^2+2x+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2x+2}$$

$$\Rightarrow 4x^2+5x+4 = (Ax+B)(x^2+2x+2)+(Cx+D)(x^2+1)$$

$$\Rightarrow 4x^2+5x+4 = (A+C)x^3+(2A+B+D)x^2+(2A+2B+C)x+2B+D$$

$$\Rightarrow A+C=0 \text{ and } 2A+B+D=4 \text{ and } 2A+2B+C=5 \text{ and } 2B+D=4$$

$$\Rightarrow A=1,\ B=2,\ C=-1, \text{ and } D=0$$

Therefore $\displaystyle\int \frac{4x^2+5x+4}{(x^2+1)(x^2+2x+2)}dx = \int \frac{x+2}{x^2+1}dx - \int \frac{x}{x^2+2x+2}dx$

$$= \int \frac{x}{x^2+1}dx + \int \frac{2}{x^2+1}dx - \int \frac{x+1-1}{x^2+2x+2}dx$$

$$= \int \frac{x}{x^2+1}dx + \int \frac{2}{x^2+1}dx - \int \frac{x+1}{x^2+2x+2}dx + \int \frac{dx}{(x+1)^2+1}$$

$$= \frac{1}{2}\ln\left|x^2+1\right| + 2\tan^{-1}x - \frac{1}{2}\ln\left|x^2+2x+2\right| + \tan^{-1}(x+1) + K$$

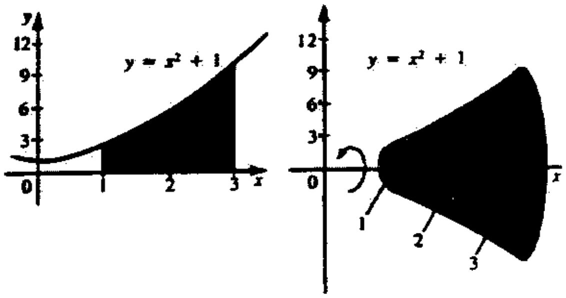
EXERCISE 11.7

1. (a) $V = \pi\int_1^3 (x^2+1)^2\,dx = \pi\int_1^3 (x^4+2x^2+1)\,dx$

$= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_1^3$

$= \pi\left[\frac{243}{5} + 18 + 3 - \frac{1}{5} - \frac{2}{3} - 1\right]$

$= \frac{1016}{15}\pi$

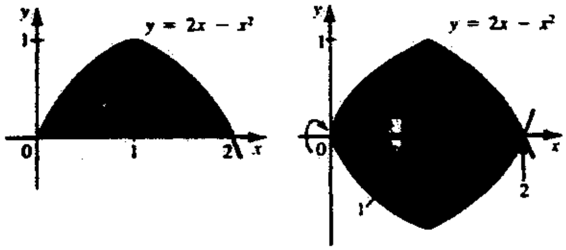


(b) The x-intercepts occur when $x(2-x) = 0$

$\Rightarrow x = 0,\ 2.$

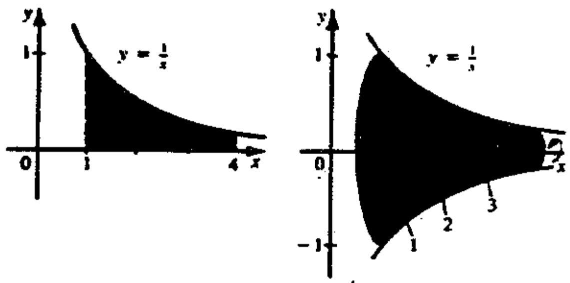
$V = \pi\int_0^2 (2x - x^2)^2\,dx = \pi\int_0^2 (4x^2 - 4x^3 + x^4)\,dx$

$= \pi\left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right]_0^2 = \pi\left[\frac{32}{3} - 16 + \frac{32}{5}\right] = \frac{16}{15}\pi$



(c) $V = \pi\int_1^4 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^4$

$= \pi\left[-\frac{1}{4} + 1\right] = \frac{3}{4}\pi$

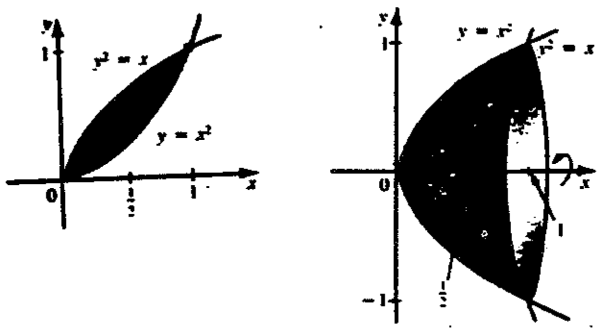


(d) The points of intersection occur when $x^2 = x^{\frac{1}{2}}$

$\Rightarrow x^2 - x^{\frac{1}{2}} = 0 \Rightarrow x^{\frac{1}{2}}(x^{\frac{3}{2}} - 1) = 0 \Rightarrow x = 0,\ 1.$

$V = \pi\int_0^1 \left[(x^{\frac{1}{2}})^2 - (x^2)^2\right]dx = \pi\int_0^1 (x - x^4)\,dx$

$= \pi\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \pi\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{10}\pi$



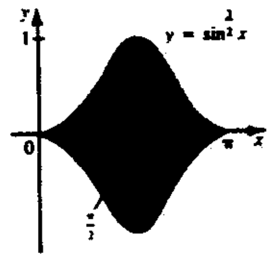
Exercise 11.7

2. (a) $V = \pi \int_0^{\pi} \sin^3 x\,dx = \pi \int_0^{\pi} (1-\cos^2 x)\sin x\,dx$

$= \pi \int_0^{\pi} \sin\, x\,dx - \pi \int_0^{\pi} \cos^2 x \sin x\,dx$

$= \pi \left[-\cos x + \frac{1}{3}\cos^3 x\right]_0^{\pi} = \pi\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$

$= \frac{4}{3}\pi$



(b) Points of intersection occur when $4 - x^2 = 3x$

$\Rightarrow x^2 + 3x - 4 = 0 \Rightarrow (x+4)(x-1) = 0$

$\Rightarrow x = -4,\ 1.$

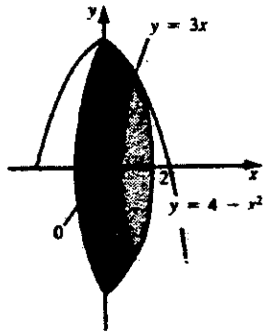
In the interval $[0,1]$:

$V = \pi \int_0^1 \left[(4-x^2)^2 - (3x)^2\right]dx$

$= \pi \int_0^1 \left[x^4 - 17x^2 + 16\right]dx$

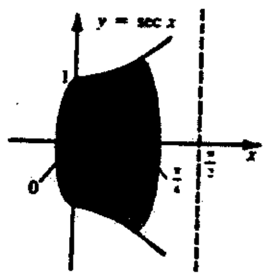
$= \pi \left[\frac{1}{5}x^5 - \frac{17}{3}x^3 + 16x\right]_0^1$

$= \pi\left[\frac{1}{5} - \frac{17}{3} + 16\right] = \frac{158}{15}\pi$



(c) $V = \pi \int_0^{0.25\pi} \sec^2 x\,dx = \pi\left[\tan x\ \right]_0^{0.25\pi}$

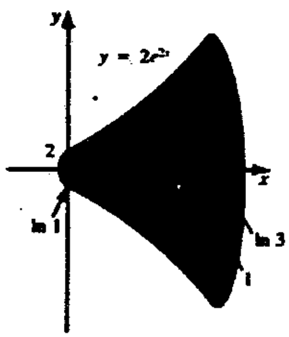
$= \pi[1 - 0\ ] = \pi$



(d) $V = \pi \int_{\ln 1}^{\ln 3} 4e^{4x}\,dx = 4\pi\left[e^{4x}\ \right]_{\ln 1}^{\ln 3}$

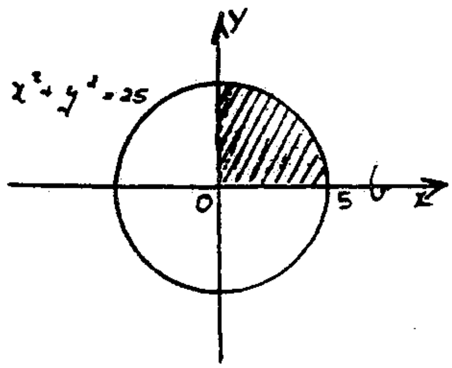
$= 4\pi\left[e^{4\ln 3} - e^{4\ln 1}\right] = 4\pi\left[3^4 - 1\ \right]$

$= 320\pi$



Exercise 11.7

3. $x^2+y^2=25 \Rightarrow y=\pm(25-x^2)^{\frac{1}{2}}$. The required volume is obtained by doubling the volume generated when the area under $y=(25-x^2)^{\frac{1}{2}}$ from 0 to 5 is rotated about the x-axis.

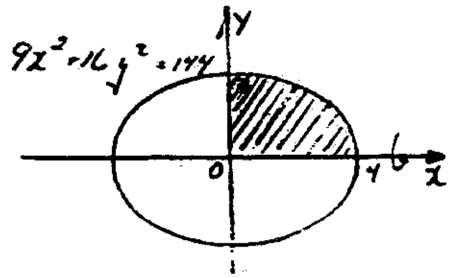


$$V=2\pi\int_0^5(25-x^2)dx=2\pi\left[25x-\frac{1}{3}x^3\right]_0^5$$

$$=2\pi\left[125-\frac{125}{3}\right]=\frac{500}{3}\pi$$

4. $9x^2+16y^2=144 \Rightarrow y=\pm\frac{1}{4}(144-9x^2)^{\frac{1}{2}}$.
The required volume is obtained by doubling the volume generated when the area under

$y=\frac{1}{4}(144-9x^2)^{\frac{1}{2}}$ from 0 to 4 is rotated about the x-axis.

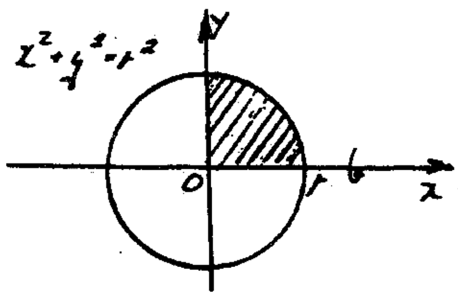


$$V=2\pi\int_0^4\frac{1}{16}(144-9x^2)dx=\frac{\pi}{8}\left[144x-\frac{9}{3}x^3\right]_0^4$$

$$=\frac{\pi}{8}\left[576-\frac{576}{3}\right]=48\pi$$

5. The sphere is generated when the circle $x^2+y^2=r^2$ is rotated about the x-axis. The required volume is obtained by doubling the volume obtained when the area

under $y=(r^2-x^2)^{\frac{1}{2}}$ from 0 to r is rotated about the x-axis.

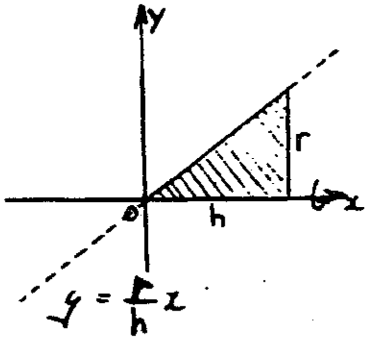


$$V=2\pi\int_0^r(r^2-x^2)dx=2\pi\left[r^2x-\frac{1}{3}x^3\right]_0^r$$

$$=2\pi\left[r^3-\frac{1}{3}r^3\right]=\frac{4}{3}\pi r^3$$

6. The cone is generated when the right triangle determined by the coordinate (h, r) is rotated about the x-axis. The hypotenuse is a segment of the line $y = \frac{r}{h}x$.

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\,dx = \frac{\pi r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h = \frac{1}{3}\pi r^2 h$$



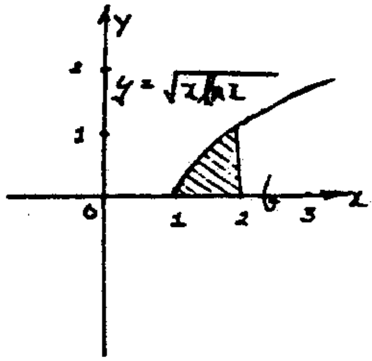
7. $$V = \pi\int_1^2 (\sqrt{x\ln x})^2\,dx = \pi\int_1^2 x\ln x\,dx$$

Let $u = \ln x$, $dv = x\,dx \Rightarrow du = \frac{dx}{x}$ and $v = \frac{1}{2}x^2$

$$uv - \int v\,du \Rightarrow \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

Therefore $\pi\int_1^2 (\sqrt{x\ln x})^2\,dx = \pi\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^2$

$$= \pi(2\ln 2 - 1 + \tfrac{1}{4}) = \pi(2\ln 2 - \tfrac{3}{4})$$



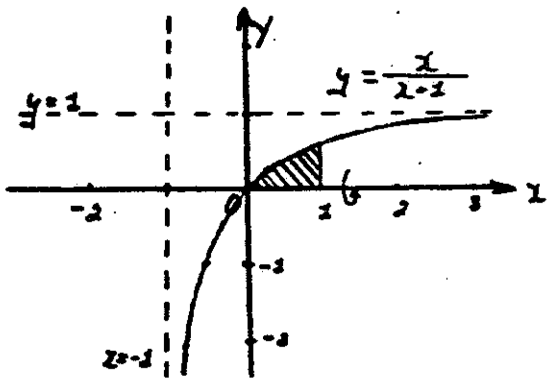
8. $$V = \pi\int_0^1\left[\frac{x}{x+1}\right]^2 dx = \pi\int_0^1 \frac{x^2}{(x+1)^2}$$

Since $\frac{x^2}{(x+1)^2} = 1 - \frac{2x+1}{x^2+2x+1}$,

$$V = \pi\int_0^1\left[1 - \frac{2x+2}{x^2+2x+1} + \frac{1}{(x+1)^2}\right]dx$$

$$= \pi\left[x - \ln(x^2+2x+1) - \frac{1}{x+1}\right]_0^1$$

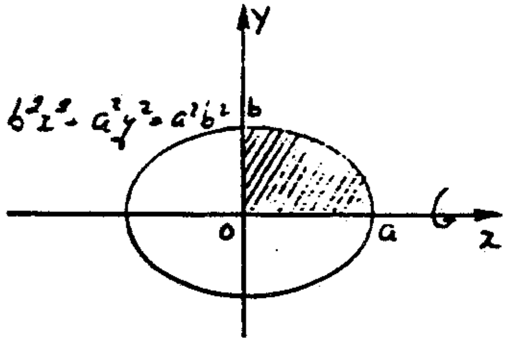
$$= \pi(1 - \ln 4 - \tfrac{1}{2} + 1) = \pi(\tfrac{3}{2} - \ln 4)$$



Exercise 11.7

9. $b^2x^2+a^2y^2=a^2b^2 \Rightarrow y=\pm\frac{1}{a}(a^2b^2-b^2x^2)^{\frac{1}{2}}$

The required volume is obtained by doubling the volume generated when the area under $y=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}$ from 0 to a is rotated about the x-axis.



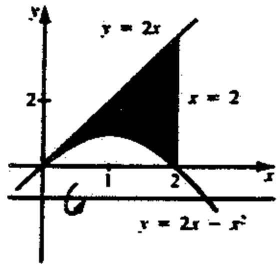
$$V=2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx=\frac{2b^2}{a^2}\pi\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

$$=\frac{2b^2}{a^2}\pi\left[a^3-\frac{1}{3}a^3\right]=\frac{2b^2}{a^2}\pi\times\frac{2}{3}a^3=\frac{4}{3}\pi ab^2.$$

In Question 3, $a=4$ and $b=3$.

Therefore $V=\frac{4}{3}(4)(3^2)\pi=48\pi$.

10 . The same volume is obtained when the region bounded by $y=2x-x^2+1$, $y=2x+1$ and $x=2$ is rotated about the x-axis.



$$V=\pi\int_0^2\left[(2x+1)^2-(2x-x^2+1)^2\right]dx$$

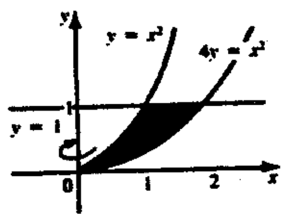
$$=\pi\int_0^2\left[-x^4+4x^3+2x^2\right]dx=\pi\left[-\frac{1}{5}x^5+x^4+\frac{2}{3}x^3\right]_0^2$$

$$=\pi\left[-\frac{32}{5}+16+\frac{16}{3}\right]=\frac{224}{15}\pi$$

11. Let y be the independent variable.

$$V=\pi\int_0^1\left[(2\sqrt{y})^2-(\sqrt{y})^2\right]dy=\pi\int_0^1 3y\,dy$$

$$=\pi\left[\frac{3}{2}y^2\right]_0^1=\frac{3}{2}\pi$$



Review Exercise 11.8

1. (a) $a = -1$, $b = 4$, so,

$\Delta x = \frac{5}{n}$, $x_i = -1 + \frac{5i}{n}$, and,

$$\int_{-1}^{4}(3x+2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{5}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(-1+\frac{5i}{n}\right)+2\right]\frac{5}{n}$$

$$= \lim_{n\to\infty}\left(-\frac{5}{n}\sum_{i=1}^{n} + \frac{75}{n^2}\sum_{i=1}^{n} i\right) = \lim_{n\to\infty}\left[-\left(\frac{5}{n}\right)n + \left(\frac{75}{n^2}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[-5 + \frac{75}{2}(1)\left(1+\frac{1}{n}\right)\right] = -5 + \frac{75}{2} = 32.5$$

(b) $a = 0$, $b = 1$, so,

$\Delta x = \frac{1}{n}$, $x_i = \frac{i}{n}$, and,

$$\int_{0}^{1}(x^3 - 2x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^3 - 2\left(\frac{i}{n}\right)^2\right]\frac{1}{n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^4}\sum_{i=1}^{n} i^3 - \frac{2}{n^3}\sum_{i=1}^{n} i^2\right) = \lim_{n\to\infty}\left[\left(\frac{1}{n^4}\right)\frac{n^2(n+1)^2}{4} - \left(\frac{2}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{4}(1)\left(1+\frac{1}{n}\right) - \frac{1}{3}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{4} - \frac{2}{3} = -\frac{5}{12}$$

2. (a) $a = -3$, $b = 0$, so,

$\Delta x = \frac{3}{n}$, $x_i = -3 + \frac{3i}{n}$, and,

$$\int_{-3}^{0}(x^2 + 2x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(-3+\frac{3i}{n}\right)^2 + 2\left(-3+\frac{3i}{n}\right)\right]\frac{3}{n}$$

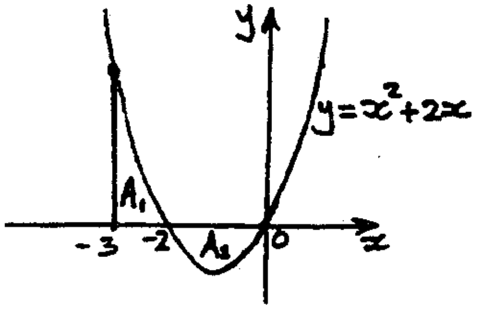
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(3 - \frac{12i}{n} + \frac{9i^2}{n^2}\right)\frac{3}{n} = \lim_{n\to\infty}\left(\frac{9}{n}\sum_{i=1}^{n} - \frac{36}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{9}{n}\right)n - \left(\frac{36}{n^2}\right)\frac{n(n+1)}{2} + \left(\frac{27}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[9 - \frac{36}{2}(1)\left(1+\frac{1}{n}\right) + \frac{27}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 9 - 18 + 9 = 0$$

(b) $\displaystyle\int_{-3}^{0}(x^2 + 2x)\,dx = \left[\frac{1}{3}x^3 + x^2\right]_{-3}^{0} = 0 - \left(-\frac{27}{3} + 9\right) = 0$

(c) $\int_{-3}^{0}(x^2 + 2x)\,dx$ **can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram. In this case, the integral is equal to zero, which implies that $A_1 = A_2$.**



3. (a) $\int (x^4 - 12x^3 + 6x)\,dx = \frac{1}{5}x^5 - 3x^4 + 3x^2 + C$

(b) $\int \sqrt{x}(1 - x + 3x^2)\,dx = \int (\sqrt{x} - x^{\frac{3}{2}} + 3x^{\frac{5}{2}})\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + \frac{6}{7}x^{\frac{7}{2}} + C$

(c) $\int (2x + \sec x \tan x)\,dx = x^2 + \sec x + C$

(d) $\int \frac{x + 2}{\sqrt[3]{x}}\,dx = \int \left(x^{\frac{2}{3}} + 2x^{-\frac{1}{3}}\right)dx = \frac{3}{5}x^{\frac{5}{3}} + 3x^{\frac{2}{3}} + C$

(e) $\int \frac{\sin x + x\cos x}{x\sin x}\,dx = \int \left(\frac{1}{x} + \frac{\cos x}{\sin x}\right)dx = \int \frac{dx}{x} + \int \cot x\,dx$

$= \ln|x| + \ln|\sin x| + C$ (see 11.3 Ex. 5. (b))

(f) Let $u = 4 + x^2$, then $du = 2x\,dx$. So,

$\int \frac{x}{\sqrt{4 + x^2}}\,dx = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C = \sqrt{4 + x^2} + C$

(g) Let $u = 1 + e^x$, then $du = e^x\,dx$. So,

$\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1 + e^x)^{\frac{3}{2}} + C$

(h) Let $u = 4x$, then $du = 4\,dx$. So,

$\int \sec 4x \tan 4x\,dx = \int \frac{1}{4}\sec u \tan u\,du = \frac{1}{4}\sec u + C = \frac{1}{4}\sec 4x + C$

(i) $u = \ln x \qquad dv = \sqrt{x}\,dx$

$du = \frac{dx}{x} \qquad v = \frac{2}{3}x^{\frac{3}{2}}$

$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}\sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$

(j) Let $u = \ln x$, then $du = \frac{dx}{x}$. So,

$\int \frac{dx}{x\ln x} = = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$

(k) Let $\frac{1}{x - x^2} = \frac{A}{x} + \frac{B}{1 - x} \Rightarrow 1 = A(1 - x) + Bx$

$\Rightarrow 1 = (-A + B)x + A$

$\Rightarrow -A + B = 0$ and $A = 1$

$\Rightarrow A = 1$ and $B = -1$.

Therefore $\int \frac{dx}{x-x^2} = \int \frac{dx}{x} - \int \frac{dx}{1-x}$

$= \ln|x| - \ln|1-x| + C = \ln\left|\frac{x}{1-x}\right| + C$

(l) Let $\frac{x+4}{x^3+3x^2-10x} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2}$

$\Rightarrow x+4 = (x^2+3x-10)A + (x^2-2x)B + (x^2+5x)C$

$\Rightarrow x+4 = (A+B+C)x^2 + (3A-2B+5C)x - 10A$

$\Rightarrow A+B+C=0$ and $3A-2B+5C=1$ and $-10A=4$

$\Rightarrow A = -\frac{2}{5}$, $B = -\frac{1}{35}$, and $C = \frac{3}{7}$.

Therefore $\int \frac{x+4}{x^3+3x^2-10x}dx = -\frac{2}{5}\int \frac{dx}{x} - \frac{1}{35}\int \frac{dx}{x+5} + \frac{3}{7}\int \frac{dx}{x-2}$

$= -\frac{2}{5}\ln|x| - \frac{1}{35}\ln|x+5| + \frac{3}{7}\ln|x-2| + C$

(m) $u = x \qquad dv = e^{-3x}dx$

$du = dx \qquad v = -\frac{1}{3}e^{-3x}$

$$\int xe^{-3x}dx = -\frac{1}{3}xe^{-3x} + \int \frac{1}{3}e^{-3x}dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

(n) $\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$

Let $u = \cos x$, then $du = -\sin x\,dx$. So,

$$\int \sin^3 x\,dx = -\int (1-u^2)du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3 x - \cos x + C$$

(o) Let $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$

$\Rightarrow x+4 = A(x+1)+B \Rightarrow x+4 = Ax+A+B$

$\Rightarrow A=1$ and $A+B=4 \Rightarrow A=1$ and $B=3$.

Therefore $\int \frac{x+4}{(x+1)^2}dx = \int \frac{dx}{x+1} + \int \frac{3\,dx}{(x+1)^2} = \ln|x+1| - \frac{3}{x+1} + C$

(p) Let $\frac{1}{x(x^2+x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x} \Rightarrow 1 = (Ax+B)x + C(x^2+x+1)$

$\Rightarrow 1 = (A+C)x^2 + (B+C)x + C$

$\Rightarrow A+C=0$ and $B+C=0$ and $C=1$

$\Rightarrow A=B=-1, C=1.$

Therefore $\int \frac{dx}{x(x^2+x+1)} = -\int \frac{x+1}{x^2+x+1}dx + \int \frac{dx}{x}$

$$= -\frac{1}{2}\int\frac{2x+1}{x^2+x+1} - \frac{1}{2}\int\frac{dx}{x^2+x+1} + \ln|x| + K_1$$

$$= -\frac{1}{2}\ln|x^2+x+1| + \ln|x| - \frac{1}{2}\int\frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} + K_2$$

$$= \ln\frac{|x|}{\sqrt{x^2+x+1}} - \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\left(x+\frac{1}{2}\right) + K$$

$$= \ln\frac{|x|}{\sqrt{x^2+x+1}} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + K$$

(q) $\frac{x^3+4x^2}{x^2+4x+3} = x - \frac{3x}{x^2+4x+3}$. Let $\frac{3x}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}$

$\Rightarrow 3x = A(x+3) + B(x+1) \Rightarrow 3x = (A+B)x + 3A + B$
$\Rightarrow A + B = 3$ and $3A + B = 0 \Rightarrow A = -\frac{3}{2}$ and $B = \frac{9}{2}$.

Therefore $\int\frac{x^3+4x^2}{x^2+4x+3}dx = \int x\,dx + \frac{3}{2}\int\frac{dx}{x+1} - \frac{9}{2}\int\frac{dx}{x+3}$

$= \frac{1}{2}x^2 + \frac{3}{2}\ln|x+1| - \frac{9}{2}\ln|x+3| + C$

(r) $\frac{x^3}{x^3+4x} = 1 - \frac{4x}{x^3+4x} = 1 - \frac{4}{x^2+4}$.

Therefore $\int\frac{x^3}{x^3+4x}dx = \int dx - \int\frac{1}{(\frac{x}{2})^2+1}dx = x - 2\tan^{-1}\left(\frac{x}{2}\right) + K$

(s) Let $u = \sin^{-1}x$, then $du = \frac{1}{\sqrt{1-x^2}}dx$. So,
$\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sin^{-1}x)^2 + C$

(t) Let $x = 3\sin\theta$, then $dx = 3\cos\theta\,d\theta$. So,

$$\int\frac{1}{(9-x^2)^{\frac{3}{2}}}dx = \int\frac{\cos\theta}{(9-9\sin^2\theta)^{\frac{3}{2}}}d\theta = \frac{1}{9}\int\frac{\cos\theta}{\cos^3\theta}d\theta = \frac{1}{9}\int\sec^2\theta\,d\theta = \frac{1}{9}\tan\theta + C$$

If $\frac{x}{3} = \sin\theta$, then $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x}{3}\right)^2}$, and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{9-x^2}}$.

Therefore, $\int\frac{1}{(9-x^2)^{\frac{3}{2}}}dx = \frac{1}{9}\int\tan\theta + C = \frac{x}{9\sqrt{9-x^2}} + C$

(u) Let $u = 1 - e^x$, then $du = -e^x dx$. So,
$\int\frac{e^x}{1-e^x}dx = -\int\frac{du}{u} = -\ln|u| + C = -\ln|1-e^x| + C$

(v) Let $u = e^x$. Therefore $\frac{du}{dx} = e^x$ and $\int \frac{e^x}{e^{2x} + 3e^x + 2} dx \Rightarrow \int \frac{du}{u^2 + 3u + 2}$.

Let $\frac{1}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1} \Rightarrow A(u+1) + B(u+2) = 1$

$\Rightarrow (A+B)u + A + 2B = 1$

$\Rightarrow A + B = 0$ and $A + 2B = 1 \Rightarrow B = 1$ and $A = -1$.

Therefore $\int \frac{du}{u^2 + 3u + 2} = \int \frac{du}{u+1} - \int \frac{du}{u+2} = \ln(u+1) - \ln(u+2) + C$ and

$$\int \frac{e^x}{e^{2x} + 3e^x + 2} dx = \ln(e^x + 1) - \ln(e^x + 2) + C$$

4. (a) $\int_{-1}^{1} (1 + 4x - x^2) dx = \left[x + 2x^2 - \frac{1}{3}x^3\right]_{-1}^{1}$

$= \left(1 + 2 - \frac{1}{3}\right) - \left(-1 + 2 + \frac{1}{3}\right) = \frac{4}{3}$

(b) $\int_0^1 e^{-3x} dx = \left[-\frac{1}{3}e^{-3x}\right]_0^1 = \frac{1}{3}(1 - e^{-3})$

(c) $\int_1^3 \frac{1 + 3x}{x^2} dx = \int_1^3 \left(\frac{1}{x^2} + \frac{3}{x}\right) dx = \left[-\frac{1}{x} + 3\ln|x|\right]_1^3$

$= \left(-\frac{1}{3} + 3\ln 3\right) - (-1) = \frac{2}{3} + 3\ln 3$

(d) Let $u = 2x + 1$, then $du = 2dx$.

When $x = 0$, $u = 1$. When $x = 4$, $u = 9$. So,

$\int_1^4 \frac{3}{2 + 1} dx = \int_1^9 \frac{3}{2u} du = \left[\frac{3}{2}\ln|u|\right]_1^9 = \frac{3}{2}\ln|9| = 3\ln 3$

(e) $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \cos x(1 - \sin^2 x)\sin^2 x\, dx$

Let $u = \sin x$, then $du = \cos x\, dx$.

When $x = 0$, $u = 0$. When $x = \frac{\pi}{2}$, $u = 1$. So,

$\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x\, dx = \int_0^1 (u^2 - u^4) du = \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

(f) $u = x^2$ $\quad dv = \sin x\, dx$

$du = 2x\, dx$ $\quad v = -\cos x$

$\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx = \left[-x^2\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2x\cos x\, dx = \int_0^{\frac{\pi}{2}} 2x\cos x\, dx$

$u = 2x$ $\quad dv = \cos x\, dx$

$du = 2dx$ $\quad v = \sin x$

Therefore, $\int_0^{\frac{\pi}{2}} x^2\sin x\, dx = \int_0^{\frac{\pi}{2}} 2x\cos x\, dx = \left[2x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\sin x\, dx$

$= \pi - \left[-2\cos x\right]_0^{\frac{\pi}{2}} = \pi - 2$

(g) $\int_0^1 \frac{x}{x+1} dx = \int_0^1 \frac{x + 1 - 1}{x + 1} dx = \int_0^1 \left(1 - \frac{1}{x+1}\right) dx = \left[x - \ln|x + 1|\right]_0^1$

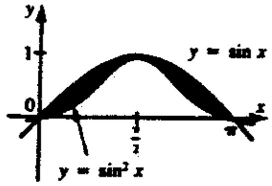
$= 1 - \ln 2$

(h) Let $x = 3\sin\theta$, then $dx = 3\cos\theta d\theta$.

When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \frac{\pi}{2}$. So,

$$\int_0^3 \sqrt{9 - x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{9 - 9\sin^2\theta}\,3\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} 9\cos^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \tfrac{9}{2}(1 + \cos 2\theta)\,d\theta$$
$$= \tfrac{9}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \tfrac{9}{2}\left(\tfrac{\pi}{2}\right) = \tfrac{9}{4}\pi$$

5. Points of intersection occur when $\sin^2 x = \sin x$



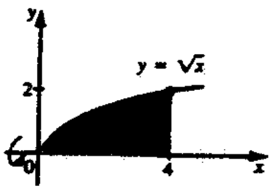
$\Rightarrow \sin x(\sin x - 1) = 0 \Rightarrow \sin x = 0$ or $\sin x = 1$

$\Rightarrow x = 0, \frac{\pi}{2}$, or π.

$$A = 2\int_0^{0.6\pi}[\sin x - \sin^2 x]dx = 2\int_0^{0.5\pi}\left[\sin x - \frac{1}{2}(1 - \cos 2x)\right]dx$$
$$= 2\left[-\cos x - \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = 2\left[-\frac{\pi}{4} + 1\right] = 2 - \frac{\pi}{2}.$$

6. (a) $V = \pi\int_0^4 x\,dx = \pi\left[\frac{1}{2}x^2\right]_0^4$

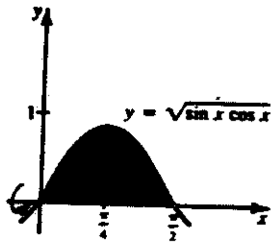
$= 8\pi$



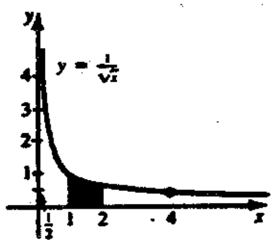
(b) $\sin x \cos x = \frac{1}{2}\sin 2x$

$$V = \pi\int_0^{0.5\pi}\left[\frac{1}{2}\sin 2x\right]dx = \pi\left[-\frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$= \frac{\pi}{4}[1 + 1] = \frac{\pi}{2}$



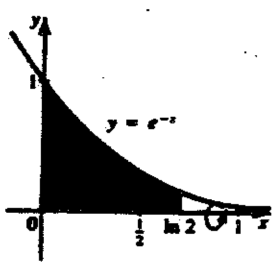
(c) $V = \pi\int_1^2 \frac{dx}{x} = \pi[\ln x]_1^2 = \pi \ln 2$



(d) $V = \pi\int_0^{\ln 2} e^{-2x}\,dx = \pi\left[-\frac{e^{-2x}}{2}\right]_0^{\ln 2}$

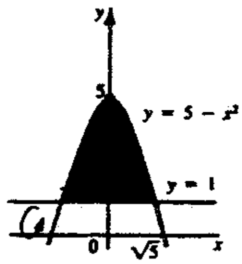
$= \pi\left[-\frac{e^{-2\ln 2}}{2} + \frac{1}{2}\right] = \pi\left[-\frac{e^{\ln\frac{1}{4}}}{2} + \frac{1}{2}\right]$

$= \pi\left[-\frac{1}{8} + \frac{1}{2}\right] = \frac{3}{8}\pi$



7. (a) The points of intersection occur when

$5-x^2=1 \Rightarrow 4=x^2 \Rightarrow x=\pm 2.$



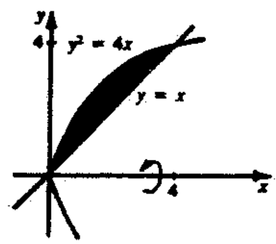
$$V=\pi\int_{-2}^{2}\left[(5-x^2)^2-(1)^2\right]dx$$

$$=2\pi\int_{0}^{2}\left[24-10x^2+x^4\right]dx$$

$$=2\pi\left[24x-\frac{10}{3}x^3+\frac{1}{5}x^5\right]_0^2=2\pi\left[48-\frac{80}{3}+\frac{32}{5}\right]=\frac{832}{15}\pi$$

(b) The points of intersection occur when $\sqrt{4x}=x$

$\Rightarrow \sqrt{x}\,(2-\sqrt{x})=0 \Rightarrow x=0 \text{ or } x=4.$



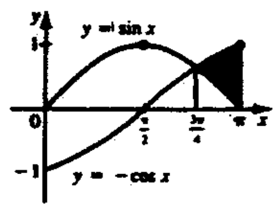
$$V=\pi\int_{0}^{4}\left[4x-x^2\right]dx=\pi\left[2x^2-\frac{1}{3}x^3\right]_0^4$$

$$=\pi\left[32-\frac{64}{3}\right]=\frac{32}{3}\pi$$

(c) $$V=\pi\int_{0.75\pi}^{\pi}\left[(-\cos x)^2-(\sin x)^2\right]dx$$

$$=\pi\int_{0.75}^{\pi}\cos 2x\,dx=\pi\left[\frac{1}{2}\sin 2x\right]_{\frac{3}{4}\pi}^{\pi}$$

$$=\pi\left[-\frac{1}{2}\sin\frac{3}{2}\pi\right]=\frac{\pi}{2}$$



Chapter 11 Test

1. (a) Define $\int_a^b f(x)\,dx$: see p. 495.

(b) $a = 0$, $b = 3$, so,

$\Delta x = \frac{3}{n}$, $x_i = \frac{3i}{n}$, and,

$$\int_0^3 (x^2 + 4x - 5)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 + 4\left(\frac{3i}{n}\right) - 5\right]\frac{3}{n}$$

$$= \lim_{n\to\infty}\left(\frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{36}{n^2}\sum_{i=1}^{n} i - \frac{15}{n}\sum_{i=1}^{n}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{27}{n^3}\right)\frac{n(n+1)(2n+1)}{6} + \left(\frac{36}{n^2}\right)\frac{n(n+1)}{2} - \left(\frac{15}{n}\right)n\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{6}(1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{36}{2}(1)\left(1 + \frac{1}{n}\right) - 15\right] = 9 + 18 - 15 = 12$$

(c) The Fundamental Theorem of Calculus: see p. 501.

(d) $$\int_0^3 (x^2 + 4x - 5)\,dx = \left[\tfrac{1}{3}x^3 + 2x^2 - 5x\right]_0^3 = 9 + 18 - 15 = 12$$

2. (a) $$\int_0^{\frac{\pi}{4}} \sin 4x\,dx = \left[-\tfrac{1}{4}\cos 4x\right]_0^{\frac{\pi}{4}} = -\tfrac{1}{4}(-1 - 1) = \tfrac{1}{2}$$

(b) $$\int_1^2 \frac{dx}{\sqrt[3]{x^4}} = \left[-3x^{-\frac{1}{3}}\right]_1^2 = 3\left(1 - 2^{-\frac{1}{3}}\right)$$

(c) $u = x$ $\quad dv = \sin x\,dx$

$du = dx$ $\quad v = -\cos x$

$$\int_0^{\frac{\pi}{2}} x\sin x\,dx = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos x\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$$

(d) Let $u = 3x + 2$, then $du = 3\,dx$.

When $x = 0$, $u = 2$. When $x = 1$, $u = 5$. So,

$$\int_0^1 \frac{1}{(3x+2)^2}\,dx = \int_2^5 \frac{du}{3u^2} = \left[-\frac{1}{3u}\right]_2^5 = -\tfrac{1}{3}\left(\tfrac{1}{5} - \tfrac{1}{2}\right) = \tfrac{1}{10}$$

3. (a) Let $u = x^3$, then $du = 3x^2\,dx$. So,

$$\int x^2 e^{x^3} = \int \tfrac{1}{3}e^u\,du = \tfrac{1}{3}e^u + C = \tfrac{1}{3}e^{x^3} + C$$

(b) $$\int \cos^5 x\,dx = \int (\cos^4 x)\cos x\,dx = \int (1 - \sin^2 x)^2\cos x\,dx$$

Let $u = \sin x$, then $du = \cos x\,dx$. So,

$$\int \cos^5 x\,dx = \int (1 - u^2)^2\,du = \int (1 - 2u^2 + u^4)\,du = u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5 + C$$

$$= \sin x - \tfrac{2}{3}\sin^3 x + \tfrac{1}{5}\sin^5 x + C$$

(c) Let $u = \sqrt{x}$, then $du = \frac{dx}{2\sqrt{x}}$. So,

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \int 2\cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$$

(d) $\int \sin^2 x\,dx = \int \frac{1}{2}(1 - \cos 2x)dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(e) Let $\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$x^2 + 1 = (x^2 - 5x + 6)A + (x^2 - 4x + 3)B + (x^2 - 3x + 2)C$

$x^2 + 1 = (A + B + C)x^2 + (-5A - 4B - 3C)x + 6A + 3B + 2C$

$A + B + C = 1$ and $-5A - 4B - 3C = 0$ and $6A + 3B + 2C = 1$

$A = 1$, $B = -5$, and $C = 5$.

Therefore $\int \frac{x^2+1}{(x-1)(x-2)(x-3)}dx = \int \frac{dx}{x-1} - 5\int \frac{dx}{x-2} + 5\int \frac{dx}{x-3}$

$= \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + K$

(f) Let $\frac{1}{x^3+3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} \rightarrow 1 = (x^2 + 3x)A + (x+3)B + Cx^2$

$1 = (A + C)x^2 + (3A + B)x + 3B \rightarrow A + C = 0$ and $3A + B = 0$ and $3B = 1$

$A = -\frac{1}{9}$, $B = \frac{1}{3}$, and $C = \frac{1}{9}$.

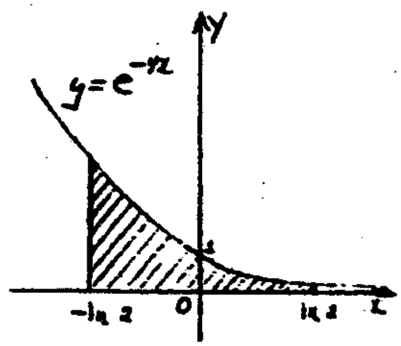
Therefore $\int \frac{dx}{x^3+3x^2} = -\frac{1}{9}\int \frac{dx}{x} + \frac{1}{3}\int \frac{dx}{x^2} + \frac{1}{9}\int \frac{dx}{x+3}$

$= -\frac{1}{9}\ln|x| - \frac{1}{3x} + \frac{1}{9}\ln|x+3| + K$

(g) $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$. Therefore $\int \frac{x^2}{x^2+1}dx = \int dx - \int \frac{dx}{x^2+1} = x - \tan^{-1}x + C$

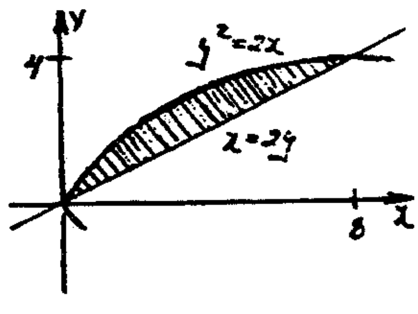
4. $V = \pi\int_{-\ln 2}^{\ln 2} e^{-4x}dx = \pi\left[-\frac{1}{4}e^{-4x}\right]_{-\ln 2}^{\ln 2}$

$= -\frac{\pi}{4}\left[e^{-4\ln 2} - e^{4\ln 2}\right] = -\frac{\pi}{4}\left[\frac{1}{16} - 16\right] = \frac{255}{64}\pi$



5. (a) $A = \int_0^8\left[\sqrt{2x} - \frac{1}{2}x\right]dx = \left[\frac{2\sqrt{2}}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2\right]_0^8$

$= \frac{2\sqrt{2}}{3} \times \sqrt{512} - 16 = \frac{64}{3} - 16 = \frac{16}{3}.$



(b) $V = \pi\int_0^8\left[2x - \frac{1}{4}x^2\right]dx = \pi\left[x^2 - \frac{1}{12}x^3\right]_0^8$

$= \pi\left[64 - \frac{512}{12}\right] = \frac{64}{3}\pi$

Problems Plus

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1. $t = \sqrt[6]{x}$ and $t \to 1$ as $x \to 1$, so $\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} = \lim_{t\to 1} \frac{t^3-1}{t^2-1}$
$= \lim_{t\to 1} \frac{(t-1)(t^2+t+1)}{(t-1)(t+1)} = \lim_{t\to 1} \frac{t^2+t+1}{t+1} = \frac{3}{2}$.

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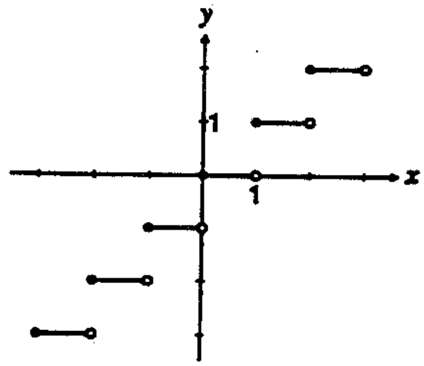
(a) For any integer n, we have

$$[\![x]\!] = n \quad \text{for} \quad n \le x < n+1$$

Therefore the graph is as shown:



(b) Since $[\![x]\!] = 2$ for $2 \le x < 3$ and $[\![x]\!] = 3$ for $3 \le x < 4$, we have

$$\lim_{x\to 3^-} [\![x]\!] = \lim_{x\to 3^-} 2 = 2 \qquad \text{and} \qquad \lim_{x\to 3^+} [\![x]\!] = \lim_{x\to 3^+} 3 = 3$$

(c) As in part (b), when n is an integer, we have

$$\lim_{x\to n^-} [\![x]\!] = n-1 \qquad \text{and} \qquad \lim_{x\to n^+} [\![x]\!] = n$$

so $\lim_{x\to n} [\![x]\!]$ does not exist. Thus $\lim_{x\to a} [\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

(d) The greatest integer function is continuous at the integers.

(e) From part (a), $[\![2x+1]\!] = n$ if $n \le 2x+1 < n+1 \Leftrightarrow n-1 \le 2x < n \Leftrightarrow \frac{n-1}{2} \le x < \frac{n}{2}$.