# REVIEW AND PREVIEW TO CHAPTER 9

**Exercise 1**

(a) $F'(x) = 16x^9 - 9x^4 + 3x$

(b) $F'(x) = 2.6(1.5x^{0.5}) - 3.7(1.9)x^{0.9} = 3.9x^{0.5} - 7.03x^{0.9}$

(c) $F'(x) = -3\left(\frac{1}{x}\right) - 5\left[\frac{1}{x^2}\right] = -\frac{3}{x} + \frac{5}{x^2}$

(d) $F'(x) = \frac{1}{2x+7}(2) + \frac{1}{2}\left(\frac{1}{\sqrt{x-3}}\right) = \frac{2}{2x+7} + \frac{1}{2\sqrt{x-3}}$

(e) $F'(x) = 6\left(-\frac{2}{x^2}\right) - 5\left(-\frac{1}{x^2}\right) + 3\left(\frac{1}{4x^2}\right)\left(4(2x)\right) = -\frac{12}{x^3} + \frac{5}{x^2} + \frac{6}{x}$

(f) $F'(x) = -\frac{1}{2}(-2\sin 2x) + 2\cos x = \sin 2x + 2\cos x$

(g) $F'(x) = 7(3\cos 3x) - 11(-7\sin 7x) = 21\cos 3x + 77\sin 7x$

(h) $F'(x) = -4\cos(x+2) + 5(-3\sin(3x-7)) = -4\cos(x+2) + 15\sin(3x-7)$

(i) $F'(x) = \frac{1}{2}(2e^{2x}) - \frac{1}{3}(-3e^{-3x}) + \frac{1}{4}(4e^{4x}) = e^{2x} + e^{-3x} + e^{4x}$

(j) $F'(x) = -5(8e^{8x}) + 2(-6e^{-6x}) = -40e^{8x} - 12e^{-6x}$

(k) $F'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{1-x}}(-1) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}}$

(l) $F(x) = \ln x + \ln(1-x)$ so $F'(x) = \frac{1}{x} + \frac{1}{1-x}(-1) = \frac{1}{x} - \frac{1}{1-x}$

(m) We do not rush into this one:

$F(x) = \ln x^4 - \ln(1-x)^5 = 4\ln x - 5\ln(1-x)$

so $F'(x) = 4\left(\frac{1}{x}\right) - 5\left(\frac{1}{1-x}\right)(-1) = \frac{4}{x} + \frac{5}{1-x}$

(n) $F'(x) = 2(e^{x^2}(2x)) - 3(e^{2x^2}(4x)) = 4xe^{x^2} - 12xe^{2x^2}$

(o) $F'(x) = \cos x\cos x + \sin x(-\sin x) = \cos^2 x - \sin^2 x$

(Using the Product Formula)

or $F(x) = \frac{1}{2}\sin 2x$ so $F'(x) = \frac{1}{2}(2\cos 2x) = \cos 2x$

(p) $F(x) = (x^2+1)^{\frac{1}{2}}$ so $F'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+1}}$

(q) $F'(x) = \frac{1}{x^3+6x+7}(3x^2+6) = \frac{3(x^2+2)}{x^3+6x+7}$

(r) $F'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

**Exercise 2**

We use the fact that if $z = a \cos kx$ then $z'' = -k^2 a \cos kx = -k^2 z$, and similarly if $w = b \sin kx$ then $w'' = -k^2 w$.

(a) $y'' = -16(3 \cos 4x) - 16(-5 \sin 4x) = -16y$

(b) $y'' = -y$

(c) $y'' = -(\sqrt{2})^2 y = -2y$

(d) $y'' = -(\sqrt{k})^2 y = -ky$

# EXERCISE 9.1

1. (a) $F(x) = x^2 + x + C$

   (b) $F(x) = x^4 - 11x + C$

   (c) $F(x) = \frac{16}{10}x^{10} - \frac{9}{5}x^5 + \frac{3}{2}x^2 + C$

   (d) $F(x) = \frac{1}{8}x^8 + \frac{1}{6}x^6 + \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$

2. (a) $F(x) = 2\left(\frac{1}{-7+1}\,x^{-7+1}\right) + \frac{1}{2}\left(\frac{1}{6}x^6\right) + C = -\frac{1}{3}x^{-6} + \frac{1}{12}x^6 + C$

   (b) $F(x) = \frac{1}{\frac{1}{2}+1}\,x^{\frac{1}{2}+1} + \frac{1}{\frac{1}{3}+1}\,x^{\frac{1}{3}+1} + C = \frac{2}{3}\,x^{\frac{3}{2}} + \frac{3}{4}\,x^{\frac{4}{3}} + C$

   (c) $F(x) = -3\ln x + 5\left(\frac{1}{-2+1}\,x^{-2+1}\right) + C = -3\ln x - 5x^{-1} + C$

   (d) $F(x) = \frac{1}{-7+1}\,x^{-7+1} + \frac{1}{-5+1}\,x^{-5+1} + \frac{1}{-3+1}\,x^{-3+1} + \ln x + C$

   $= -\frac{1}{6}x^{-6} - \frac{1}{4}x^{-4} - \frac{1}{2}x^{-2} + \ln x + C$

3. (a) $F(x) = \ln(-x) + C$

   (b) $F(x) = 2\left(\frac{1}{-3+1}\,x^{-3+1}\right) - 3\left(\frac{1}{-2+1}\,x^{-2+1}\right) + C$

   $= -x^{-2} + 3x^{-1} + C$

   (c) $F(x) = \frac{1}{\frac{3}{2}}(-x)^{\frac{3}{2}}(-1) + C = -\frac{2}{3}(-x)^{\frac{3}{2}} + C$

   (d) $F(x) = \frac{1}{-4+1}\,x^{-4+1} + \frac{1}{3+1}\,x^{3+1} + \frac{1}{-2+1}\,x^{-2+1} + C$

   $= -\frac{1}{3}x^{-3} + \frac{1}{4}x^4 - x^{-1} + C$

4. (a) $F(x) = -\frac{1}{2}\cos 2x + 2\sin x + C$

   (b) $F(x) = -\frac{3}{5}\sin 5x - 8\cos x + C$

   (c) $F(x) = 7\sin x - 11\left(-\frac{1}{11}\cos 11x\right) + C = 7\sin x + \cos 11x + C$

(d) $F(x) = -4 \sin(x + 2) + C$

5. (a) $F(x) = e^{x} - e^{-x} + C$

(b) $F(x) = e^{x} + e^{-x} + C$

(c) $F(x) = 4(\frac{1}{2}e^{2x}) - 6(-\frac{1}{3}e^{-3x}) + C = 2e^{2x} + 2e^{-3x} + C$

(d) $F(x) = e^{x} - (-\frac{1}{2}e^{-2x}) + \frac{1}{3}e^{3x} + C = e^{x} + \frac{1}{2}e^{-2x} + \frac{1}{3}e^{3x} + C$

6. (a) $F(x) = \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} - \frac{1}{\frac{1}{2}+1}(1 - x)^{\frac{1}{2}+1}(-1) + C = \frac{2}{3}\left[x^{\frac{3}{2}} + (1 - x)^{\frac{3}{2}}\right] + C$

(b) $F(x) = \ln x + \ln(1 - x) + C = \ln(x - x^{2}) + C$

(c) $F(x) = \frac{1}{(-\frac{1}{2}+1)}(1 - x)^{-\frac{1}{2}+1}(-1) + \frac{1}{(-\frac{1}{2}+1)} x^{-\frac{1}{2}+1} + C$

$= -2\sqrt{1 - x} + 2\sqrt{x} + C = 2(\sqrt{x} - \sqrt{1 - x}) + C$

(d) $F(x) = 4 \ln x - 5 \ln(1 - x) + C$

7. (a) $F(x) = \frac{1}{2}e^{x^{2}} + C$

(b) $F(x) = \frac{1}{3}\sin^{3}x + C$

(c) $F(x) = \ln(x^{2} + 1) + C$

(d) $F(x) = \sqrt{x^{2} + 1} + C$

8. (a) $F(x) = \arctan x + C$

(b) $f(x) = -\frac{\sin x}{\cos x} = \frac{1}{\cos x}\frac{d}{dx}\cos x$, so

$F(x) = \ln \cos x + C$

(c) $F(x) = \sec x + C$

(d) $f(x) = e^{\ln x} = x$ for all $x > 0$, so

$F(x) = \frac{1}{2}x^{2} + C$

9. From Solution 2, $F(x) = -\frac{1}{2}\cos^{2}x + C_2 = -\frac{1}{2}\left(\frac{\cos 2x + 1}{2}\right) + C_2$

$= -\frac{1}{4}\cos 2x - \frac{1}{4} + C_2$

Now solution 3 is $F(x) = -\frac{1}{4}\cos 2x + C_3$ so $C_3 = C_2 - \frac{1}{4}$.

10. (a) Since $\frac{d}{dx} \ln|x| = \frac{1}{x}$ for all $x \neq 0$, we have $F'(x) = \frac{1}{x}$ for $x > 0$, and $F'(x) = \frac{1}{x}$ for $x < 0$. So $F'(x) = \frac{1}{x}$ for all $x \neq 0$.

(b) If $F(x) = \ln|x| + C$ then $F(1) = \ln 1 + C = c$. So $c = 3$ from the definition of $f(x)$. Hence $F(x) = \ln|x| + 3$. Then $F(-1) = \ln 1 + 3 = 3$. But $F(-1) = \ln|-1| - 7 = -7$. The contradiction that $3 = -7$ shows that it is impossible for $F(x) = \ln|x| + C$ no matter what $c$ is.

(c) We have $F'(x) = \frac{d}{dx} \ln|x|$ yet $F(x) \neq \ln|x| + C$ from part (b).

(d) The statement about the most general antiderivative only applies on an interval: $(-\infty, 0) \cup (0, \infty)$ is not an interval.

# EXERCISE 9.2

1. If $y' = 4x - 3$ then $y = 2x^2 - 3x + C$. So we substitute the initial conditions to determine C.
   (a) $0 = 2(0)^2 - 3(0) + C$ Thus $C = 0$, and $y = 2x^2 - 3x$
   (b) $-1 = 2(0)^2 - 3(0) + C$ Thus $C = -1$, and $y = 2x^2 - 3x - 1$
   (c) $2 = 2(-1)^2 - 3(-1) + C$ Thus $C = -3$, and $y = 2x^2 - 3x - 3$
   (d) $0 = 2(3)^2 - 3(3) + C$ Thus $C = -9$, and $y = 2x^2 - 3x - 9$

2. (a) $s = 4.9t^2 + C$, so $0 = 4.9(0)^2 + C$. Hence $s = 4.9t^2$
   (b) $s = \frac{1}{4}t^4 - \frac{1}{2}t^2 + C$, so $0 = C$. Hence $s = \frac{1}{4}t^4 - \frac{1}{2}t^2$
   (c) $s = -\cos t + C$, so $0 = -\cos 0 + C$. Hence $s = -\cos t + 1$
   (d) $s = \frac{1}{0.1}e^{0.1t} + C$, so $0 = 10e^0 + C$. Hence $s = 10(e^{0.1t} - 1)$

3. (a) $F(x) = x^3 - x^2 + 6x + C$. Now $F(2) = 3$, so $3 = 2^3 - 2^2 + 6(2) + C$. Thus $C = -13$, and $F(x) = x^3 - x^2 + 6x - 13$.

   (b) $F'(x) = 3\sqrt{2}\sqrt{x}$ so $F(x) = 3\sqrt{2}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C$. $3 = 3\sqrt{2}\left(\frac{2}{3}\right)\left(2\right)^{\frac{3}{2}} + C$. Thus $C = 3 - \sqrt{2}\,(2)\,(2\sqrt{2}) = -5$, and $F(x) = 2\sqrt{2}\,x^{\frac{3}{2}} - 5$.

   (c) $F(x) = 4e^{\frac{x}{2}} + C$. So $3 = 4e^{\frac{2}{2}} + C$, and so $C = 3 - 4e$. Hence $F(x) = 4e^{\frac{x}{2}} + 3 - 4e$.

   (d) $F(x) = \frac{2}{3}(x^{\frac{3}{2}} + (4 - x)^{\frac{3}{2}}) + C$. So $3 = \frac{2}{3}(2^{\frac{3}{2}} + 2^{\frac{3}{2}}) + C$. Hence $C = 3 - \frac{8\sqrt{2}}{3}$, and $F(x) = \frac{2}{3}(x^{\frac{3}{2}} + (4 - x)^{\frac{3}{2}}) + 3 - \frac{8\sqrt{2}}{3}$.

4. (a) $y = \sin x - \cos x + C$. Now (0,0) is on the curve, so $0 = \sin 0 - \cos 0 + C$. Thus $C = 1$ and $y = \sin x - \cos x + 1$.
   (b) $y = e^x - e^{-x} + C$. So $0 = e^0 - e^0 + C$, and thus $C = 0$. Hence $y = e^x - e^{-x}$.
   (c) $y = 2\sqrt{x + 1} + C$. So $0 = 2\sqrt{1} + C$ and $C = -2$. Hence $y = 2(\sqrt{x + 1} - 1)$.
   (d) $y = \frac{1}{4}(x^2 + 1)^2 + C$. So $0 = \frac{1}{4}(0^2 + 1) + C$, and $C = -\frac{1}{4}$. Hence $y = \frac{1}{4}[(x^2 + 1)^2 - 1] = \frac{1}{4}x^4 + \frac{1}{2}x^2$.

OR:
$\frac{dy}{dx} = x^3 + x$ so $y = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$. So $0 = C$ and $y = \frac{1}{4}x^4 + \frac{1}{2}x^2$.

5. If the line $x + y = 0$ is tangent to a graph, then at the point of tangency the slope must be $-1$. So we find x such that $F'(x) = -1$, and thus determine the point(s) of tangency.

(a) $F'(x) = x$. So $x = -1$ for $F'(x) = 1$. Next $F(x) = \frac{1}{2}x^2 + C$. If $x = -1$ then $(-1,1)$ is the point of tangency so $1 = F(-1) = \frac{1}{2}(-1)^2 + C$ and thus $C = \frac{1}{2}$. Hence $F(x) = \frac{1}{2}(x^2 + 1)$.

(b) If $F'(x) = x^3$, then $(-1,1)$ is the point of tangency. Now $F(x) = \frac{1}{4}x^4 + C$ so $1 = F(-1) = \frac{1}{4}(-1)^4 + C$ and thus $C = \frac{3}{4}$. Hence $F(x) = \frac{x^4 + 3}{4}$.

(c) If $F'(x) = -x^5$ then $(1,-1)$ is the point of tangency. Now $F(x) = -\frac{1}{6}x^6 + C$ so $-1 = F(1) = -\frac{1}{6} + C$ and thus $C = -\frac{5}{6}$. Hence $F(x) = -\frac{1}{6}(x^6 + 5)$.

(d) If $F'(x) = -1$ then every point is a possible point of tangency. Now $F(x) = -x + C$. If $C \neq 0$, then $y = -x + C$ is parallel to, but distinct from $x + y = 0$, so these lines are not tangent. So we must have $C = 0$. Hence $F(x) = -x$.

6. For $y = e^x$, $\frac{dy}{dx} = e^x$. So we look for $y = F(x)$ with $F'(x) = e^x$. Hence $F(x) = e^x + C$ $\quad \lim_{x\to\infty} e^x + C = \infty$ so no horizontal asymptotes at $\infty$. Next $\lim_{x\to-\infty}(e^x + C) = 0 + C = C$. So $C = 4$ and $y = F(x) = e^x + 4$ is the desired equation.

7. If $F'(x) = -x^2$ then $F(x) = -\frac{1}{3}x^3 + C$. Now $y = F(x)$ has two tangents of slope $-1$; $x + y = 0$, and $x + y = \frac{4}{3}$. The only places $F'(x) = 1$ are at $x = 1$ and $x = -1$. The points of tangency are $(1, C - \frac{1}{3})$, and $(-1, C + \frac{1}{3})$. We now compute $x + y$ for both these points: $C + \frac{2}{3}$, $C - \frac{2}{3}$. So $C + \frac{2}{3} = 0$, $C - \frac{2}{3} = \frac{4}{3}$ or $C + \frac{2}{3} = \frac{4}{3}$, $C - \frac{2}{3} = 0$. Hence $C = \frac{2}{3}$ since the first pair of equations have no solution. Hence $F(x) = \frac{1}{3}(2 - x^3)$.

8. (a) $\frac{ds}{dt} = \cos 2t$, so $s = \frac{1}{2}\sin 2t + C$. Thus $1 = \frac{1}{2}\sin 2(\frac{\pi}{4}) + C$, and so $C = \frac{1}{2}$. Hence $s = \frac{1}{2}(1 + \sin 2t)$.

(b) $\frac{ds}{dt} = \frac{1 - \cos 2t}{2} = \frac{1}{2} - \frac{1}{2}\cos 2t$. So $s = \frac{1}{2}t - \frac{1}{4}\sin 2t + C$. Now $s = 0$ when $t = \frac{\pi}{2}$ so $0 = \frac{1}{2}(\frac{\pi}{2}) - \frac{1}{4}\sin 2(\frac{\pi}{2}) + C$

$$0 = \frac{\pi}{4} + C.$$

Hence $C = -\frac{\pi}{4}$ and thus $s = \frac{1}{2}t - \frac{1}{4}\sin 2t - \frac{\pi}{4}$.

# EXERCISE 9.3

1. Recall that our convention is that $s = 0$ when $t = 0$. Since $v = \frac{ds}{dt} = 6t - 3t^2$, $s = 3t^2 - t^3$.
   (a) When $t = 1$, $s = 3(1)^2 - 1^3 = 2$.
   (b) When $t = 2$, $s = 3(2)^2 - 2^3 = 4$.
   So the object moves 4 m in the first 2 seconds. Note that distance equals displacement as $\frac{ds}{dt} > 0$ for $0 < t < 2$.
   (c) When $t = 3$, $s = 0$ so the object is back at the start. Since $v = 0$ only at $t = 2$ for $0 < t < 3$ the object turned back at $t = 2$.
   So it travelled $2 \times 4 = 8$ m in all.

2. Let $v$ denote the velocity of the canister $t$ seconds after it is dropped. Then $v = 9.8t$ so $s = 4.9t^2$, where $s$ is the distance dropped in that time. The canister hits when $s = 500$ so $500 = 4.9t^2$, and thus $t \doteq 10.1$. The impact velocity is $9.8(10.1) \doteq 99$. Since $99 < 100$ it probably won't burst.

3. From Question 7 we have $h = -4.9t^2 - 8t + 63$ where we use the same variables as in Example 3. We have to find $t$ such that $h = 0$.
   So $0 = 4.9t^2 + 8t - 63$. Thus
   $$t = \frac{-8 + \sqrt{8^2 - 4(4.9)(-63)}}{9.8} \doteq 2.9.$$
   It will take approximately 2.9 s.

4. From Question 7, $h = -4.9t^2 + 5t + 80$. We find $t$ such that $h = 0$. So $4.9t^2 - 5t - 80 = 0$,
   $$t = \frac{5 + \sqrt{5^2 - 4(4.9)(-80)}}{9.8}$$
   which is approximately 4.6.

5. Here $v = at$, so $s = \frac{1}{2}at^2 + c$. But $s = 0$ when $t = 0$ so $s = \frac{1}{2}at^2$. We find $a$ by noting that $v = 30$ when $t = \frac{1}{6}$, so $a = 180$. Hence $s = \frac{1}{2}(180)(\frac{1}{6})^2 = 2.5$. [Note that the units for velocity are km/h, so $t$ is measured in h, so 10 min $= \frac{1}{6}$h.]

**Exercise 9.3**

6. Since acceleration is the rate of change of velocity we have $\frac{dv}{dt} = -0.9t + 9$ so $v = -0.45t^2 + 9t + c$. The initial velocity is 10, so $v = -0.45t^2 + 9t + 10$. This is valid for $0 \leq t \leq 10$. Next we have $s = -0.15t^3 + 4.5t^2 + 10t$ (since $s = 0$ when $t = 0$).
   (a) Thus when $t = 10$, $s = -0.15(10)^3 + 4.5(10)^2 + 10(10)$. The raindrop falls 400 m.
   (b) The velocity does not change after $t = 10$ since the acceleration is zero. Now when $t = 10$, $v = -0.45(10)^2 + 9(10) + 10 = 55$.
   (c) Since the drop falls 400 m in the first 10 seconds we need to find how long it takes to drop 200 m more at 55 m/s. This takes about 3.6 s. So the whole fall takes $10 + 3.6 = 13.6$ s.

7. We use the variables of Example 3. First $v = -9.8t + C$, so $v = -9.8t + v_0$, since the initial velocity is $v_0$ m/s. [Note: By our convention on h being measured "up," velocity is positive if the object goes up, and negative if it is thrown down]. Since $v = \frac{dh}{dt}$ we have $h = -4.9t^2 + v_0t + C$. Now $h = h_0$ when $t = 0$ so $h = -4.9t^2 + v_0t + h_0$.

   We find t such that $h = 0$: $4.9t^2 - v_0t - h_0 = 0$

$$t = \frac{v_0 + \sqrt{v_0^2 - 4(4.9)(-h_0)}}{9.8}$$

$$= \frac{v_0 + \sqrt{v_0^2 + 19.6h_0}}{9.8}$$

The other solution to this quadratic is a negative number. Since $t \geq 0$, it is extraneous.

# EXERCISE 9.4

1. Let V denote the value of the certificate at time t years. Then $\frac{dV}{dt} = 0.0625V$ since interest is compounded continuously. Then $V = Ce^{0.0625t}$. Now $V = 1000$ when $t = 0$, so $V = 1000e^{0.0625t}$. When $t = 5$, $V = 1000e^{0.0625(5)} \doteq 1366.84$.

2. Let P denote the number of people in the town. The $\frac{1}{P}\frac{dP}{dt} = 0.016$, so $P = Ce^{0.016t}$. Say the population of the town is $P_0$ initially, then $P = P_0e^{0.016t}$. We have to find t such that $2P_0 = P_0e^{0.016t}$. Thus $0.016t = \ln 2$ and $t \doteq 43a$.

3. Let G, B denote the number of grey and black moths, at time t (in months). Then $\frac{1}{G}\frac{dG}{dt} = 0.03$ and $\frac{1}{B}\frac{dB}{dt} = 0.04$. Suppose there are initially $B_0$ black and $G_0$ grey. Then $B = B_0e^{0.04t}$, $G = G_0e^{0.03t}$. We also know that $G_0 = 2B_0$. So we have $B = B_0e^{0.04t}$, $G = 2B_0e^{0.03t}$. We want t such that $B = 2G$: $B_0e^{0.04t} = 2(2B_0e^{0.03t})$. So $e^{0.04t} = 4e^{0.03t}$ and $e^{0.01t} = 4$. Thus $t = \frac{\ln 4}{0.01} \doteq 139$. In about 11.5 a the tables will be turned.

4. We use $T = A + (T_0 - A)e^{kt}$, where $T_0 = 100$, $A = 10$. So $T = 10 + 90e^{kt}$. When $t = 1$, $T = 25$ so $25 = 10 + 90e^{k}$, and thus $e^k = \frac{1}{6}$. Hence $T = 10 + 90(\frac{1}{6})^t$.
   (a) Solve: $50 = 10 + 90(\frac{1}{6})^t$, so $6^t = \frac{90}{40}$, $t = \frac{\ln \frac{9}{4}}{\ln 6} \doteq 0.45$ s.
   (b) Solve $12 = 10 + 90(6^{-t})$, so $6^t = \frac{90}{2} = 45$. So $t = \frac{\ln 45}{\ln 6} \doteq 2.1$.

5. We have $T = 23 + (-7 - 23)e^{kt}$. When $t = \frac{1}{2}$, $T = 8$, so $8 = 23 - 30e^{k(\frac{1}{2})}$. Thus $e^{\frac{1}{2}k} = \frac{1}{2}$ so $\frac{1}{2}k = \ln(\frac{1}{2})$, $k = -2\ln 2 = \ln(\frac{1}{4})$. Hence $T = 23 - 30(4^{-t})$. So when $t = 3$, $T = 23 - 30(4^{-3}) \doteq 22.5°$ C.

6. Here $T = 24 + (175 - 24)e^{kt}$. When $t = 4$, $T = 175 - 60 = 115$, so $115 = 24 + 151e^{k(4)}$. Hence $k = \frac{1}{4}\ln(\frac{151}{91}) \doteq -0.13$. Thus $T = 24 + 151e^{-0.13t}$. Now we find t such that $37 = 24 + 151e^{-0.13t}$. So $e^{0.13t} = \frac{151}{13}$, hence $t = \frac{1}{0.13}\ln(\frac{151}{13}) \doteq 19$ min.

7. $T = 35 + (T_0 - 35)e^{kt}$, since $A = 35$.
$29 = 35 + (T_0 - 35)e^{k(1)}$, so $(35 - T_0)e^{k} = 6$ 1
$32 = 35 + (T_0 - 35)e^{k(2)}$, so $(35 - T_0)e^{2k} = 3$ 2
We square 1: $(35 - T_0)^2 e^{2k} = 36$
We multiply 2 by $35 - T_0$ : $(35 - T_0)^2 e^{2k} = (35 - T_0)3$.
Hence $36 = 3(35 - T_0)$, $12 = 35 - T_0$, $T_0 = 23°C$.

8. (a) $\frac{dA}{dt} = k(M - A)$

(b) Since $\frac{d(M - A)}{dt} = -\frac{dA}{dt} = -k(M - A)$ we have $M - A = Ce^{-kt}$.
Now $A = 0$ when $t = 0$ so $M - 0 = Ce^0$, and so $C = M$. Thus
$M - A = Me^{-kt}$ and hence $A = M(1 - e^{-kt})$.

(c) We infer that for Jim $M = 40$. So $A = 40(1 - e^{-kt})$. Next, when $t = 5$,
$A = 13$ so $13 = 40(1 - e^{-k(5)})$. Hence $e^{-5k} = 1 - \frac{13}{40}$, $-5k = \ln(\frac{27}{40})$.
So $k \doteq 0.08$. This gives $A = 40(1 - e^{-0.08t})$ as the relationship between $A$ and $t$. We now find $t$ such that $A = 36$:
$36 = 40(1 - e^{-0.08t})$, so $e^{-0.08t} = 0.1$
$t = \frac{1}{-0.08} \ln 0.1 \doteq 29$ min.

9. We are told that $L\frac{dI}{dt} = E - RI = R(\frac{E}{R} - I)$, which looks like Newton's law of cooling.
So $\frac{dI}{dt} = \frac{R}{L}(\frac{E}{R} - I)$ and thus $\frac{d(\frac{E}{R} - I)}{dt} = -\frac{dI}{dt} = -\frac{R}{L}(\frac{E}{R} - I)$

Hence $\frac{E}{R} - I = Ce^{-\frac{R}{L}t}$. Since $I = 0$ when $t = 0$, $C = \frac{E}{R}$. Thus

$I = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$.

# EXERCISE 9.5

1. Consult Question 8 for the derivation of the formula:

$$A = cV + (A_0 - cV)e^{-\frac{r}{V}t}$$

Here $A_0 = 20$ kg, $V = 1000$L, $c = 100 \times 10^{-3} = 10^{-1}$ kg/L, and $r = 8$ L/min. So

$$A = 100 + (20 - 100)e^{-0.008t}$$

(a) When $t = 60$, $A = 100 - 80e^{-0.008(60)} \doteq 50.5$ kg

(b) Find $t$ when $A = 80$: $80 = 100 - 80e^{-0.008t}$ so $e^{0.008t} = 4$, and thus $t = \frac{1}{0.008}\ln 4 \doteq 173$ min.

2. Here $A_0 = 20$, $V = 1000$, $c = 10^{-1}$ and $r = 20$ so $A = 100 - 80e^{-0.02t}$

(a) We $t = 60$, $A = 100 - 80e^{-0.02(60)} \doteq 75.9$ kg.

(b) Find $t$ when $A = 80$: $80 = 100 - 80e^{-0.02t}$ so $e^{0.02t} = 4$, and thus $t = \frac{1}{0.02}\ln 4 \doteq 69$ min.

3. Here $A_0 = 20$, $V = 1000$, $c = 0.1$ and $r = 8$. So $A = 100 - 95e^{-0.008t}$.

(a) When $t = 60$, $A = 100 - 95e^{-0.008(60)} \doteq 41.2$ kg.

(b) Find $t$ when $A = 80$: $80 = 100 - 95e^{-0.008t}$ so $e^{-0.008t} = \frac{20}{95}$ and thus $t = \frac{1}{0.008}\ln\left(\frac{95}{20}\right) \doteq 195$.

4. Here $A_0 = 20$, $V = 500$, $c = 0.1$, $r = 8$.

So $A = (0.1)500 + (20 - (0.1)500)e^{-\frac{8}{500}t} = 50 - 30e^{-0.016t}$

(a) When $t = 60$, $A = 50 - 30e^{-0.016(60)} \doteq 38.5$ kg.

5. The differential equation for this process is $\frac{dA}{dt} = k(30 - A)$, since $30 - A$ is the amount of undissolved sugar.

Thus $\frac{d}{dt}(30 - A) = -\frac{dA}{dt} = -k(30 - A)$, and therefore $30 - A = Ce^{-kt}$.

Since $A = 0$ when $t = 0$ we have $A = 30(1 - e^{-kt})$.

When $t = 5$, $A = 25$ so we can find k: $25 = 30(1 - e^{-k(5)})$ so $e^{-5k} = \frac{1}{6}$ and thus $k = \frac{1}{5}\ln 6 \doteq 0.36$.

(a) $A = 30(1 - e^{-0.36t})$

(b) Find $t$ such that $A = 15$. This gives $e^{-0.36t} = \frac{1}{2}$ so $t = \frac{1}{0.36}\ln 2 = 2$ min.

(c) We find $t$ such that $29 = 30(1 - e^{-0.36t})$. So $e^{-0.36t} = \frac{1}{30}$ and thus $t = \frac{1}{0.36}\ln 30 \doteq 10$.

## Exercise 9.5

6. Let M kg denote his mass at time t days. Now $\Delta M$ = [mass in] — [mass out] over the time interval $\Delta t$. First we look at [cal in] = $1280\Delta t$.

   Check units: $\frac{\text{cal}}{\text{d}} \times \text{d} = \text{cal}$.

   Next [cal out] = $16\ M\ \Delta t$

   Check units: $\frac{\text{cal}}{\text{kg}\cdot\text{d}} \times \text{kg} \times \text{d} = \text{cal}$

   Now cal $= \frac{1}{10\,000}$ kg so $\Delta M = (\frac{1280}{10\,000} - \frac{16M}{10\,000})\Delta t$

   $\frac{dM}{dt} = \frac{16}{10\,000}(80 - M)$

   Hence, in the usual way $80 - M = Ce^{-\frac{16}{10\,000}t}$

   (a) Thus $M = 80 + 15e^{-0.0016t}$, since his initial mass is 95 kg.

   (b) $\lim_{t\to\infty} M = 80$ kg

   (c) Find t such that $M = 80 + \frac{1}{2}(95 - 80) = 87.5$

   $87.5 = 80 + 15e^{-0.0016t}$. Thus $e^{0.0016t} = 2$ and so $t = \frac{1}{0.0016}\ln 2 \doteq 433$.

7. $\frac{d}{dt}(mv) = gm$ gives $\frac{dm}{dt}v + m\frac{dv}{dt} = gm$. But $\frac{dm}{dt} = km$ so $kmv + m\frac{dv}{dt} = gm$ and thus $\frac{dv}{dt} = g - kv = k(\frac{g}{k} - v)$.

   Hence $\frac{d(\frac{g}{k} - v)}{dt} = -\frac{dv}{dt} = -k(\frac{g}{k} - v)$ and we deduce that $\frac{g}{k} - v = Ce^{-kt}$.

   This gives: $v = \frac{g}{k} - Ce^{-kt}$

   Now $\lim_{t\to\infty} v = \frac{g}{k} - C\lim_{t\to\infty} e^{-kt} = \frac{g}{k} - 0$ since $\lim_{t\to\infty} e^{-kt} = 0$ as k is positive.

   The terminal velocity is $\frac{g}{k}$. (Check units: $\frac{m}{s^2} \times \frac{1}{\frac{1}{s}} = m/s$ — velocity)

8. We follow Example 1 in our discussion of $\Delta A$ = [salt in] — [salt out]

   Now [salt in] = [volume in] × [conc. of salt]

   = [rate in] × $\Delta t$ × [conc. of salt]

   $= r \times \Delta t \times c$

   Check units: $\frac{\text{L}}{\text{min}} \times \text{min} \times \frac{\text{kg}}{\text{L}} = \text{kg}$.

   [salt out] = [rate out] × $\Delta t$ × [conc. of salt]

   $= r \times \Delta t \times \frac{A}{V}$

   So $\Delta A \doteq (rc - r\frac{A}{V})\Delta t$

   Now $\frac{\Delta A}{\Delta t} \doteq \frac{r}{V}(cV - A)$

   So, taking the limit as $\Delta t \to 0$

   $\frac{dA}{dt} = \frac{r}{V}(cV - A)$, $\frac{d}{dt}(cV - A) = -\frac{dA}{dt} = -\frac{r}{V}(cV - A)$,

   so $cV - A = \text{const}\ e^{-\frac{r}{V}t}$. Now $A = A_0$ when $t = 0$.

   $cV - A = (cV - A_0)e^{-\frac{r}{V}t}$; $A = cV + (A_0 - cV)e^{-\frac{r}{V}t}$.

# EXERCISE 9.6

1. Assuming logistic growth we have $P = \frac{6000}{1 + 19e^{-kt}}$, since $K = 6000$, and the initial population is

   $300 = \frac{6000}{1 + 19e^{0}}$.

   We determine the growth rate k from the fact that when $t = 1$, $P = 2 \times 300$:

   $600 = \frac{6000}{1 + 19e^{-k(1)}}$, so $1 + 19e^{-k} = 10$ and thus $k = \ln(\frac{19}{9}) = 0.75$.

   (a) $P = \frac{6000}{1 + 19e^{-0.75t}}$

   (b) When $t = 4$, $P = \frac{6000}{1 + 19e^{-3}} \doteq 3080$

   (c) We find t so that $P = 4800$:

   $4800 = \frac{6000}{1 + 19e^{-0.75t}}$, so $1 + 19e^{-0.75t} = \frac{6000}{4800}$

   Hence $e^{0.75t} = 76$, and so $t = \frac{1}{0.75} \ln 76 \doteq 5.8$ a

   [Remark: exponential growth would give $t = 4$, since the population doubles in a year 300, 600, 1200, 2400, 4800]

2. Here we have $P = \frac{700}{1 + 69e^{-kt}}$ since $K = 700$, and $10 = \frac{700}{1 + 69e^{0}}$. When $t = 9$, $P = 500$ so we can determine k:

   $500 = \frac{700}{1 + 69e^{-9k}}$ so $69e^{-9k} = \frac{700}{500} - 1$.

   Thus $e^{9k} = 69(\frac{5}{2})$ and $k \doteq \frac{1}{9}\ln(172.5) \doteq 0.57$. We have $P = \frac{700}{1 + 69e^{-0.57t}}$.

   To find when 350 cells were present we find t such that

   $1 + 69e^{-0.57t} = 2$ or $e^{0.57t} = 69$. Thus $t = \frac{1}{0.57}\ln 69 \doteq 7.4$ h.

3. We apply the logistic function with $K = 2000$: $P = \frac{2000}{1 + Ce^{-kt}}$. Since $P = 10$ when $t = 0$ we have $1 + Ce^{0} = 200$, so $C = 199$.

   Hence $P = \frac{2000}{1 + 199e^{-kt}}$ Now when $t = 1$, $P = 100$ so $1 + 199e^{-k(1)} = 20$,

   thus $e^{k} = \frac{199}{19}$. We see that $k = \ln(\frac{199}{19}) \doteq 2.35$, and deduce

   $P = \frac{2000}{1 + 199e^{-2.35t}}$.

(a) Half the town is infected when $1 + 199e^{-2.35t} = 2$, so $e^{2.35t} = 199$.
Thus $t = \frac{1}{2.35}\ln 199 \doteq 2.25$; about 16 days.

(b) When $t = 2.25 + 1$, $P = \frac{2000}{1 + 199e^{-2.35(3.26)}} \doteq 1825$. So about 800 new cases appear.

4. We begin with $P = \frac{K}{1 + Ce^{-kt}}$, so that $K = P(1 + Ce^{-kt})$. We measure time in decades starting at 1950 which is $t = 0$.

$K = 14(1 + Ce^{0})$ $\quad C = \frac{K}{14} - 1$ $\quad$ 1

$K = 18.2(1 + Ce^{-k})$ $\quad Ce^{-k} = \frac{K}{18.2} - 1$ $\quad$ 2

$K = 21.6(1 + Ce^{-2k})$ $\quad Ce^{-2k} = \frac{K}{21.6} - 1$ $\quad$ 3

We eliminate $e^{-k}$ and $e^{-2k}$ by comparing 1 × 3 and 2 × 2 :

$C(Ce^{-2k}) = (\frac{K}{14} - 1)(\frac{K}{21.6} - 1)$ $\quad$ 4

$C^2e^{-2k} = (\frac{K}{18.2} - 1)^2$ $\quad$ 5

Now 4 − 5 gives $(\frac{K}{14} - 1)(\frac{K}{21.6} - 1) = (\frac{K}{18.2} - 1)^2$

so $\frac{K^2}{302.4} - K(\frac{1}{14} + \frac{1}{21.6}) + 1 = \frac{K}{331.24} - K(\frac{2}{18.2}) + 1$

Hence $\frac{K}{302.4} + \frac{36.6}{302.4} = \frac{K}{331.24} - \frac{1}{9.1}$

$K(\frac{1}{302.4} - \frac{1}{331.24}) = \frac{36.6}{302.4} - \frac{1}{9.1}$

$K \doteq 27.2$. The limiting population is 27.2 million.

Now from 1 $C = \frac{27.2}{14} - 1 = 0.94$, and from

2 : $0.94e^{-k} = \frac{27.2}{18.2} - 1 \doteq 0.4945$; thus $k = \ln(\frac{0.94}{0.494})$ and so $k = 0.64$.

Our logistic function for Canada's population is

$P = \frac{27.2}{1 + 0.94e^{-0.64t}}$. Fits when $t = 2$.

(b) Now let $t = 3$: $P = 23.9$

(c) Next find t such that $25 = \frac{27.2}{1 + 0.94e^{-0.64t}}$. Thus $1 + 0.94e^{-0.64t} = \frac{27.2}{25}$

and so $e^{-0.64t} = \frac{1}{0.94} \times 0.088$

$t = -\frac{1}{0.64}\ln(\frac{0.088}{0.94}) \doteq 3.7$

Our formula predicts 3.7 decades from 1950, that is 1987.

5. Let C denote cigarette consumption per capita, and let t measure decades since 1900. Then $C = \frac{4000}{1 + 79e^{-kt}}$ fits the limiting consumption of 4000, and the initial consumption of

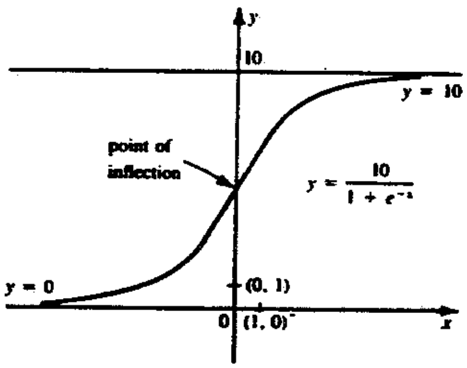
$50 = \frac{4000}{1 + 79e^0}$. To determine k we use t = 6.

$$3900 = \frac{4000}{1 + 79e^{-6k}}, \text{ so } 79e^{-6k} = \frac{1}{39}$$

Hence $e^{-k} = \frac{1}{(79 \times 39)^{\frac{1}{6}}} \doteq 0.262$. Thus $C = \frac{4000}{1 + 79(0.262)^t}$

| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| C | 50 | 185 | 620 | 1650 | 2915 | 3645 | 3900 | 3975 |

6. If $y = \frac{10}{1 + e^{-x}}$ then $\frac{1}{y}\frac{dy}{dx} = \frac{1}{1}(1 - \frac{y}{10})$ since y satisfies the logistic equation with K = 10 and k = 1. Hence $\frac{dy}{dx} = y(1 - \frac{y}{10}) = y = \frac{1}{10}y^2$. Since $y > 0$, and $y < 10$, $\frac{dy}{dx} > 0$ for all x. There are no maxima and minima. Of course y = 0, and y = 10 are horizontal asymptotes. Finally, we consider concavity. Since $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dy}(\frac{dy}{dx})\frac{dy}{dx} = (1 - \frac{2}{10}y)(y - \frac{1}{10}y^2)$ we have $\frac{d^2y}{dx^2} > 0$ for $0 < y < 5$ and $\frac{d^2y}{dx^2} > 0$ for $5 < y < 10$. There is an inflection point at y = 5, which corresponds to x = 0. The slope at the inflection point is $5(1 - \frac{5}{10}) = 2.5$. Here is a sketch.




7. Since $Q = kP - \frac{k}{K}P^2$, $\frac{dQ}{dP} = k - \frac{2k}{K}P$

Thus $\frac{dQ}{dt} = \frac{dQ}{dP}\frac{dP}{dt} = k(1 - \frac{2P}{K})kP(1 - \frac{P}{K})$. Since $0 < P < K$, $\frac{dQ}{dt} > 0$ for $2P < K$ and $\frac{dQ}{dt} < 0$ for $2P > K$. Thus Q is increasing then decreasing. So Q has a maximum at $2P = K$, so $P = \frac{K}{2}$.

This can be seen without calculus through noting that

$\frac{1}{k}\frac{Q}{K} = \frac{1}{K}P - \frac{1}{K^2}P^2 = \frac{1}{4} - (\frac{1}{2} - \frac{P}{K})^2$ by completing the square. So $\frac{1}{kK}Q \leq \frac{1}{4}$.

The maximum value of Q is $\frac{k}{4}K$.

# EXERCISE 9.7

1. The solution of the differential equation is $s = A\cos 2t + B\sin 2t$, which gives $\frac{ds}{dt} = -2A\sin 2t + 2B\cos 2t$.

   At $t = 0$, $s = A$ and $\frac{ds}{dt} = 2B$, so $A = s\,|_{t=0}$ and $B = \frac{1}{2}\frac{ds}{dt}\,|_{t=0}$

   (a) $s = (0)\cos 2t + (\frac{1}{2}\times 0)\sin 2t = 0$

   (b) $s = (1)\cos 2t + (\frac{1}{2}\times 0)\sin 2t = \cos 2t$

   (c) $s = (-1)\cos 2t + (\frac{1}{2}\times 2)\sin 2t = -\cos 2t + \sin 2t$

   (d) $s = (3)\cos 2t + (\frac{1}{2}\times -5)\sin 2t = 3\cos 2t - 2.5\sin 2t$

2. (a) $y = A\cos x + B\sin x$

   (b) $y = A\cos 3x + B\sin 3x$

   (c) Since $y'' + \frac{9}{4}y = 0$ we have $\sqrt{k} = \sqrt{\frac{9}{4}} = 1.5$

   so $y = A\cos 1.5x + B\sin 1.5x$

   (d) $y = A\cos\sqrt{2}x + B\sin\sqrt{2}x$

3. Since $(0,1)$ is on the graph $f(0) = 1$, and since $2x + y = 1$ is tangent there, $f'(0) = -2$. Now $f(x) = A\cos\sqrt{k}x + B\sin\sqrt{k}x$ if $f'' + kf = 0$ so $1 = f(0) = A$, and $-2 = f'(0) = B\sqrt{k}$. Hence $A = 1$ and $B = -\frac{2}{\sqrt{k}}$. Thus $f(x) = \cos\sqrt{k}x - \frac{2}{\sqrt{k}}\sin\sqrt{k}x$.

   (a) $f(x) = \cos x - 2\sin x$ since $\sqrt{k} = \sqrt{1} = 1$

   (b) $f(x) = \cos 2x - \sin 2x$ since $\sqrt{k} = \sqrt{4} = 2$

   (c) $f(x) = \cos 1.5x - \frac{2}{1.5}\sin 1.5x$ since $\sqrt{k} = \sqrt{2.25} = 1.5$

   (d) $f(x) = \cos\sqrt{2}x - \frac{2}{\sqrt{2}}\sin\sqrt{2}x$ since $\sqrt{k} = \sqrt{2}$

4. The general equation is $\frac{d^2s}{dt^2} + \frac{k}{m}s = 0$. Here, as in Example 3, $\frac{k}{m} = 40$ so $\frac{d^2s}{dt^2} + 40s = 0$. Thus $s = A\cos\sqrt{40}t + B\sin\sqrt{40}t$ and

   $\frac{ds}{dt} = -A\sqrt{40}\sin\sqrt{40}t + B\sqrt{40}\cos\sqrt{40}t$.

   So $\frac{ds}{dt}\,|_{t=0} = B\sqrt{40}$, $s\,|_{t=0} = A$.

   Recall also that $s$ is measured up. In this question the mass is released from rest, so $\frac{ds}{dt}\,|_{t=0} = 0$, and so $B = 0$. Thus $s = A\cos\sqrt{40}t$, $A = s\,|_{t=0}$

   (a) Since $A = 0.6 - 0.7 = -0.1$ we have $s = -0.1\cos\sqrt{40}t$.

(b) Since $A = 0.60 - 0.42 = 0.18$ we have $s = 0.18 \cos \sqrt{40}t$.

(c) Since $A = 0.60 - 1.00 = -0.40$ we have $s = -0.40 \cos \sqrt{40}t$.

(d) Since $A = 0.60 - 0.21 = 0.39$ we have $s = 0.39 \cos \sqrt{40}t$.

5. See solution to Question 4 for the general treatment. Here $A = 0.60 - 0.86 = -0.26$ and $B = \frac{1}{\sqrt{40}} \frac{ds}{dt} \Big|_{t = 0}$.

(a) $s = -0.26 \cos \sqrt{40}t + \frac{1}{\sqrt{40}} \sin \sqrt{40}t$

(b) $s = -0.26 \cos \sqrt{40}t + \frac{-2}{\sqrt{40}} \sin \sqrt{40}t$

(c) $s = -0.26 \cos \sqrt{40}t + \frac{3.7}{\sqrt{40}} \sin \sqrt{40}t$

(d) $s = -0.26 \cos \sqrt{40}t + \frac{-4.1}{\sqrt{40}}$ son $\sqrt{40}t$

6. The point of this question is to discover that if $f(x) = A \cos \ell x + B \sin \ell x$, then the maximum value of $f$ is $\sqrt{A^2 + B^2}$ and the minimum value is $-\sqrt{A^2 + B^2}$. This gives the maximum displacement for simple harmonic motion. If $f(x) = A \cos x + B \sin x$ then $f'(x) = -A \sin x + B \cos x$. So $f$ has extreme values when $A \sin x = B \cos x$ that is $\sin x = \frac{\pm B}{\sqrt{A^2 + B^2}}$ and $\cos x = \frac{\pm A}{\sqrt{A^2 + B^2}}$. From this the result follows.

(a) $\sqrt{1^2 + 1^2} = \sqrt{2}$ (b) $\sqrt{1^2 + (\sqrt{3})^2} = 2$

(c) $\sqrt{3^2 + (-4)^2} = 5$ (d) $\sqrt{(-2)^2 + 1} = \sqrt{5}$

Another method is to write $A \cos \ell x + B \cos \ell x = \sqrt{A^2 + B^2} \cos (\ell x - n)$ for a suitable n. Then the result is obvious. For example, we have:

(a) $f(x) = \sqrt{2} \cos (x - \frac{\pi}{4})$(b) $f(x) = 2 \cos (x - \frac{\pi}{3})$

7. We first determine k in Hooke's Law. $k(0.70 - 0.53) = 4.25$, so $k = 25$

Since $m = 1$, the differential equation for the displacement is $\frac{d^2s}{dt^2} + \frac{25}{1}s = 0$. Hence $s = A \cos 5t + B \sin 5t$.

Now $A = 0.53 - 0.66 = -0.13$, and $\frac{ds}{dt}\Big|_{t = 0} = -2.1$ so $B = -\frac{2.1}{5} = -0.42$.

Thus $s = -0.13 \cos 5t - 0.42 \sin 5t$. So the maximum displacement is $\sqrt{(-0.13)^2 + (-0.42)^2} \doteq 0.44$ m.

# 9.8 REVIEW EXERCISE

1. (a) $F(x) = 3(\frac{1}{2}x^2) - \pi(x) + C = 1.5x^2 - \pi x + C$

   (b) $F(x) = e(-\cos x) + \sqrt{2}(\sin x) + C = -e\cos x + \sqrt{2}\sin x + C$

   (c) $F(x) = 4(\frac{1}{\sqrt{2}}e^{\sqrt{2}x}) - \frac{1}{7}(-\frac{1}{\pi}e^{-\pi x}) + C = 2\sqrt{2}e^{\sqrt{2}x} + \frac{1}{7\pi}e^{-\pi x} + C$

   (d) $F(x) = \ln(x^4 + 1) + C$

2. (a) $F(x) = \frac{1}{10}\ln x + \sqrt{2}(\frac{1}{-2+1}x^{-1}) + C$

   $= 0.1\ln x - \sqrt{2}x^{-1} + C$

   (b) $F(x) = 4(\frac{1}{1.5+1}x^{2.6}) - 3(\frac{1}{2.7+1}x^{3.7}) + C$

   $= 1.6x^{2.5} - \frac{3}{3.7}x^{3.7} + C$

   (c) $f(x) = \sqrt{x} + \sqrt{2}\sqrt{x} + \sqrt{3}\sqrt{x} = (1 + \sqrt{2} + \sqrt{3})\sqrt{x}$

   $F(x) = (1 + \sqrt{2} + \sqrt{3})\frac{2}{3}x^{\frac{3}{2}} + C.$

   (d) $F(x) = e^{\frac{1}{x}} + C$

3. (a) $F(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 + C$ but $4 = F(-1) = \frac{2}{3}(-1)^3 - \frac{3}{2}(-1)^2 + C$

   so $C = 4 + \frac{13}{6} = \frac{37}{6}$. Hence $F(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 + \frac{37}{6}$

   (b) $F(x) = e^x + \frac{1}{2}e^{-2x} + C$ but $4 = F(-1) = e^{-1} + \frac{1}{2}e^2 + C$

   so $C = 4 - e^{-1} - \frac{1}{2}e^2$. Hence $F(x) = e^x + \frac{1}{2}e^{-2x} + 4 - e^{-1} - \frac{1}{2}e^2$.

   (c) $F(x) = -\cos x - \sin x + C$ but $4 = -\cos(-1) - \sin(-1) + C$

   so $C = 4 + \cos 1 - \sin 1$. Thus $F(x) = -\cos x - \sin x + 4 + \cos 1 - \sin 1$.

   (d) $F(x) = \frac{1}{3}(3 + 2x)^{\frac{3}{2}} + C$, but $4 = \frac{1}{3}(3 + 2(-1))^{\frac{3}{2}} + C$

   so $C = 4 - \frac{1}{3} = \frac{11}{3}$. Hence $F(x) = \frac{1}{3}(3 + 2x)^{\frac{3}{2}} + \frac{11}{3}$.

4. Look back to Exercise 9.3 Question 7 to see that $h = -4.9t^2 + v_0 t + h_0$, and the time taken to reach the ground is given by $t = \frac{v_0 + \sqrt{v_0^2 + 19.6h_0}}{9.8}$.

   In this case $v_0 = 30$, $h_0 = 210$. So $t = \frac{30 + \sqrt{900 + 4116}}{9.8} \doteq 10.3$ s.

5. $\frac{dv}{dt} = a = 8.4 - 0.7t$ so $v = 8.4t - 0.35t^2 + C$. At $t = 0, v = 13$ so $13 = 8.4(0) - 0.35(0)^2 + C$. Hence $\frac{ds}{dt} = v = 8.4t - 0.35t^2 + 13$.
So $s = 4.2t^2 - \frac{0.35}{3}t^3 + 13t + C$. Initially $s = 0$ so $c = 0$:
$s = 13t + 4.2t^2 - \frac{0.35}{3}t^3$.
   (a) When $t = 12$, $s = 559.2$
   (b) When $t = 12$, $v = 8.4(12) - 0.35(12)^2 + 13 = 63.4$
   (C) It takes 12 s to fall 559.2 m. At 63.4 m/s it takes $\frac{1000 - 569.2}{63.4} \doteq 7$ to fall the remainder of the way. So total drop takes about $12 + 7 = 19$ seconds.

6. $T = A + (T_0 - A)e^{kt}$. Here $T_0 = 105$, $A = 17$ so $T = 17 + 88e^{kt}$. When $t = 1$, $T = 37$ so $37 = 17 + 88e^k = \frac{20}{88} = \frac{5}{22}$ Thus, when $T = 2$, $T = 17 + 88(\frac{5}{22})^2 \doteq 21.5 \doteq 22^\circ$ C.

7. Look back to Question 8, Exercise 9.5 $A = CV + (A_0 - cV)e^{-\frac{r}{v}t}$.
Here $A_0 = 10$, $V = 800$, $c = 75 \times 10^{-3}$, $r = 24$
so $A = 60 + (10 - 60)e^{-0.03t} = 60 - 50e^{-0.03t}$.
   (a) When $t = 60$, $A = 60 - 50e^{-1.8} \doteq 52$.
   (b) If $A = 35$ then $t$ is given by solving $35 = 60 - 50e^{-0.03t}$; so $e^{-0.03t} = 2$ and thus $t = \frac{\ln 2}{0.03} \doteq 23$.

8. We use $P = \frac{K}{1 + Ce^{-kt}}$. Here $K = 420$ and $20 = \frac{420}{1 + C}$ so $C = 20$.

Thus $P = \frac{420}{1 + 20e^{-kt}}$. Next, when $t = 1$, $P = 160$ so $160 = \frac{420}{1 + 20e^{-k}}$.

Hence $20e^{-k} = \frac{420}{160} - 1$ and so $e^{-k} = 0.08125$. This gives $k \doteq 2.5$.
   (a) So we obtain the formula $P = \frac{420}{1 + 20e^{-2.5t}}$.
   (b) When $t = 3$, $P = \frac{420}{1 + 20e^{-7.5}} \doteq 415$.
   (c) Find $t$ such that $P = 220$. So solve $220 = \frac{420}{1 + 20e^{-2.5t}}$:
   $20e^{-2.5t} = \frac{420}{220} - 1$; $e^{-2.5t} = 0.0454$. Thus $t \doteq 1.24$ d.

9. $y = A\cos 5x + B\sin 5x$, so $y' = -5A\sin 5x + 5B\cos 5x$

(a) $0 = A\cos 0 + B\sin 0$ so $A = 0$.

$-3 = -5A\sin 0 + 5B\cos 0$, so $B = -0.6$

Thus $y = -0.6\sin 5x$

(b) $2 = A\cos 0 + B\sin 0$ so $A = 2$

$1 = -5A\sin 0 + 5B\cos 0$ so $B = 0.2$

Thus $y = 2\cos 5x + 0.2\sin 5x$.

(c) $-1 = A\cos 5\pi + B\sin 5\pi$, so $A = 1$.

$0 = -5A\sin 5\pi + 5B\cos 5\pi$, so $B = 0$.

Thus $y = \cos 5x$

(d) $3 = A\cos 10\pi + B\sin 10\pi$, so $A = 3$.

$3 = -5A\sin 10\pi + 5B\cos 10\pi$, so $B = 0.6$.

Thus $y = 3\cos 5x + 0.6\sin 5x$.

10. Since $\frac{d^2s}{dt^2} = -3s$ we have $\frac{d^2s}{dt^2} + 3s = 0$ so $s = A\cos\sqrt{3}t + B\sin\sqrt{3}t$. Since $s = 0$ when $t = 0$ we have $A = 0$. Thus $s = B\sin\sqrt{3}t$. Hence $\frac{ds}{dt} = B\sqrt{3}\cos\sqrt{3}t$. Since $\frac{ds}{dt}\Big|_{t=0} = 4$ we have $B\sqrt{3} = 4$. So $B = \frac{4}{\sqrt{3}}$.

(a) Hence $s = \frac{4}{\sqrt{3}}\sin\sqrt{3}t$.

(b) Max value of $s$ is $\frac{4}{\sqrt{3}}$ since $\left|\sin\sqrt{3}t\right| \leq 1$ for al t.

# CHAPTER 9 TEST

1. (a) F is an antiderivative of f if, and only if, $F'(x) = f(x)$ for each x in the interval.

   (b) $F(x) = \frac{1}{3}x^3 + 3(\frac{1}{-1}e^{-x}) + 4(-\cos x) + c$
   $= \frac{1}{3}x^3 - 3e^{-x} - 4\cos x + c$

2. Since $F'(x) = \sqrt{2}x^{\frac{1}{2}} + 6$ we have $F(x) = \sqrt{2}\left(\frac{1}{\frac{1}{2}+1}\right)x^{\frac{3}{2}} + 6x + c.$
   So $F(x) = \frac{2\sqrt{2}}{3}x^{\frac{3}{2}} + 6x + c$. Now $F(2) = 5$,
   so $5 = \frac{2\sqrt{2}}{3}x^{\frac{3}{2}} + 12 + c$ and thus $c = 5 - \frac{8}{3} - 12 = -\frac{29}{3}$.
   Hence $F(x) = \frac{1}{3}(2\sqrt{2}x^{\frac{3}{2}} + 18x - 29)$.

3. $t = \frac{v_0 + \sqrt{v_0^2 + 19.6h_0}}{9.8}$ from Exercise 9.3 Q7. So here $v_0 = 25$, $h = 46$.
   Thus $t \doteq 6.5$.

4. We use $T = A + (T_0 - A)e^{kt}$, with $A = 22$, $T_0 = 80$ to obtain $T = 22 + 58e^{kt}$.
   When $t = 1$, $T = 60$ so $60 = 22 + 58e^k$. Thus $e^k = \frac{38}{58}$ and so
   $k = \ln(\frac{38}{58}) \doteq -0.42$. Hence $T = 22 + 58e^{-0.42t}$ We now find t such that
   $T = 33$: $\quad 33 = 22 + 58e^{-0.42t}$ $\quad$ Thus $e^{-0.42t} = \frac{11}{58}$ and so
   $t = \frac{1}{0.42}\ln(\frac{58}{11}) \doteq 4$. So after $4 - 1 = 3$ more minutes the coffee reaches 33° C.

5. Recall $A = cV + (A_0 - cV)e^{-\frac{r}{V}t}$ from Exercise 9.5. Here $c = 0$
   (fresh water in) so $A = A_0e^{-\frac{r}{V}t} = 1e^{-\frac{15}{100}t} = e^{-0.15t}$ since
   $A_0 = 100 \times 10 \times 10^{-3} = 1$, and $r = 15$, $V = 100$. When is $A = \frac{1}{2}$?
   When $0.15t = \ln 2$, so $t = \frac{\ln 2}{0.15} \doteq 4.6$.

6. (a) $\frac{dN}{dt} = kN(R - N)$ where $N$ is the number who have heard the rumour, and $R$ is the number of rumour prone people.

(b) Here $R = 1500$, so $N = \frac{1500}{1 + ce^{-kt}}$. When $t = 0$, $N = 6$ so $1 + c = \frac{1500}{6} = 250$. Thus $c = 249$, which gives $N = \frac{1500}{1 + 249e^{-kt}}$. When $t = 3$, $N = 750$ so $1 + 249e^{-3k} = 2$. This gives $e^{3k} = 249$, so $k = 1.8$. The solution is $N = \frac{1500}{1 + 249e^{-1.8t}}$.

7. Since $y'' + \frac{4}{9}y = 0$ we have $y = A\cos\frac{2}{3}x + B\sin\frac{2}{3}x$. At $x = 0$, $y = 2$, so $2 = A\cos 0 + B\sin 0$. This gives $A = 2$. So $y = 2\cos\frac{2}{3}x + B\sin\frac{2}{3}x$. Since $y' = -\frac{4}{3}\sin\frac{2}{3}x + \frac{2B}{3}\cos\frac{2}{3}x$. Since $y' = -3$ when $x = 0$ we have

$$-3 = -\frac{4}{3}\sin 0 + \frac{2B}{3}\cos 0$$

This gives $B = -4.5$. Hence $y = 2\cos\left(\frac{2x}{3}\right) - 4.5\sin\left(\frac{2x}{3}\right)$.