CALCULUS
A FIRST COURSE

THE McGRAW-HILL RYERSON
MATHEMATICS PROGRAM

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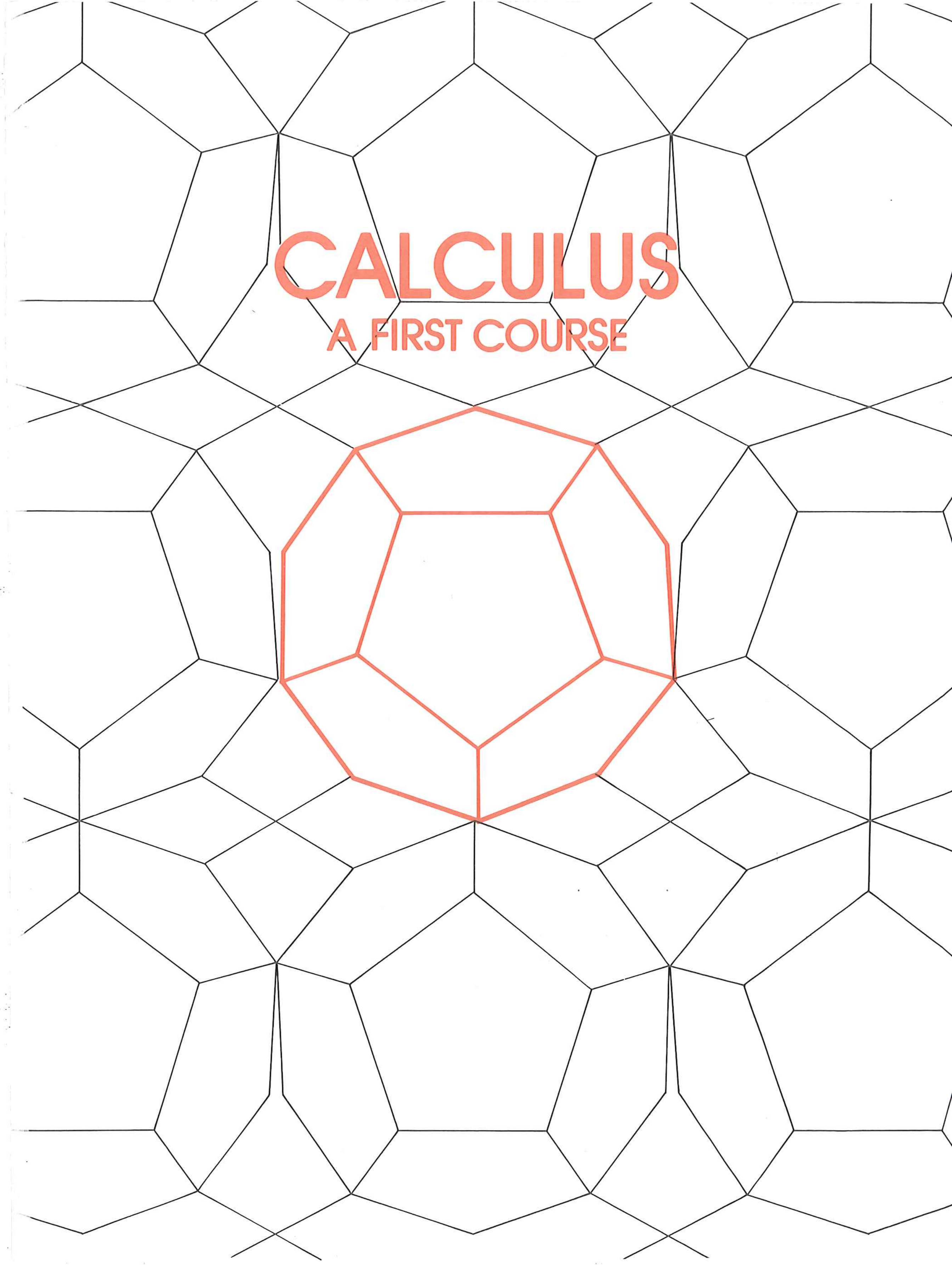

THE McGRAW-HILL RYERSON MATHEMATICS PROGRAM

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FINITE MATHEMATICS
ALGEBRA AND GEOMETRY
CALCULUS: A FIRST COURSE

CALCULUS
A FIRST COURSE

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PREFACE

This textbook on Calculus is part of a three-volume series, also including books on Finite Mathematics and Algebra and Geometry, for courses that represent the culmination of a high school mathematics program.

APPLICATIONS

Most students need the motivation of realistic applications to learn calculus. We have selected a diverse range of applications from the physical, social, and engineering sciences, as well as from mathematics itself. Included are the following:

- We show how derivatives occur as the slope of a tangent, the velocity of a car, the linear density of a wire, the rate of growth of an animal or bacteria population, the rate of change of temperature, the rate of flow of water, the rate of spread of an epidemic, the rate of reaction in chemistry, and the marginal cost and marginal profit in economics.
- We show how to minimize the cost of laying cable across a river, the cost of fencing a field, and the average cost of producing a commodity. We show how to maximize revenue or profit if cost and demand functions are known.
- We explain the radiocarbon dating of ancient objects.
- We show how Newton's Law of Cooling can be used to find the temperature of a 900°C rod of steel after it has been cooled by forced air.
- We solve a differential equation to find the number of fish in a lake at a given time.

PROBLEM SOLVING EMPHASIS

Our educational philosophy has been strongly influenced by the books of George Polya and the lectures of both Polya and Gabor Szego at Stanford University. They consistently introduced a topic by relating it to something concrete or familiar. In this spirit, we have tried to motivate new topics by relating mathematical concepts to the students' experiences.

The influence of Polya's work on problem solving can be seen throughout the book. The Review and Preview to Chapter 3 gives an introduction to some of the problem-solving strategies that he has explained at greater length in his books *How to Solve It, Mathematical Discovery*, and *Mathematics and Plausible Reasoning*. When these strategies occur in examples, we highlight their use with margin captions.

In addition to the graded exercise sets, we have included special problems, called PROBLEMS PLUS, that require a higher level of problem-solving skill.

ILLUSTRATIONS/ANSWERS

We have included an unusually large amount of art in order to convey the notion of change that is basic to calculus. The answer section alone contains 384 diagrams, many of them answers to the curve-sketching questions. All answers are given at the end of the text. Complete solutions to every question are available in the *Solutions Manual*.

FOUNDERS OF CALCULUS

We have included biographies of five mathematicians who played a major role in the invention and advancement of calculus: Sir Isaac Newton, Gottfried Leibniz, Pierre Fermat, the Bernoulli family, and Leonhard Euler. We believe that an account of the historical development of calculus helps to make the subject come alive.

ACKNOWLEDGMENTS

In addition to the reviewers listed earlier and our consultant John Carter, who attended all our authors' meetings, we wish to thank our teaching colleagues for their valuable advice, the editorial and production staff at McGraw-Hill Ryerson for a superb job, and those close to us who understandingly put up with the long hours that we devoted to this project.

James Stewart
Thomas M.K. Davison
Bryan Ferroni

CHAPTER 1

LIMITS AND RATES OF CHANGE

REVIEW AND PREVIEW TO CHAPTER 1

Factoring

Example 1 Factor $x^2 - 3x - 18$.

Solution The two integers that add to give -3 and multiply to give -18 are -6 and 3. Therefore

$$x^2 - 3x - 18 = (x - 6)(x + 3)$$

Some special polynomials can be factored using the following formulas.

$a^2 - b^2 = (a - b)(a + b)$	(difference of squares)
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	(difference of cubes)
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	(sum of cubes)

Example 2 Factor. (a) $x^3 + 27$ (b) $2x^4 - 18x^2$

Solution (a) Using the formula for a sum of cubes with $a = x$ and $b = 3$, we have

$$x^3 + 27 = x^3 + 3^3 = (x + 3)(x^2 - 3x + 9)$$

(b) $2x^4 - 18x^2 = 2x^2(x^2 - 9)$ (common factor)
$= 2x^2(x - 3)(x + 3)$ (difference of squares)

The Factor Theorem

A polynomial $P(x)$ has $x - b$ as a factor if and only if $P(b) = 0$.

Example 3 Factor $P(x) = 2x^3 - 5x^2 - 4x + 3$.

Solution

$$P(1) = 2(1)^3 - 5(1)^2 - 4(1) + 3 = -4 \neq 0$$
$$P(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3 = 0$$

Therefore, by the Factor Theorem, $x + 1$ is a factor. We find another factor by long division:

$$\begin{array}{rl}
 & 2x^2 - 7x + 3 \\
x + 1 & \overline{)\, 2x^3 - 5x^2 - 4x + 3} \\
 & \underline{2x^3 + 2x^2} \\
 & -7x^2 - 4x \\
 & \underline{-7x^2 - 7x} \\
 & 3x + 3 \\
 & \underline{3x + 3}
\end{array}$$

Thus we have

$$\begin{aligned}
P(x) &= 2x^3 - 5x^2 - 4x + 3 \\
&= (x + 1)(2x^2 - 7x + 3) \\
&= (x + 1)(2x - 1)(x - 3)
\end{aligned}$$

When factoring expressions that involve fractional or negative exponents, we use the Laws of Exponents.

Example 4 Factor $2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$.

Solution The term with the smallest exponent is $-6x^{-\frac{1}{2}}$ and we use $2x^{-\frac{1}{2}}$ as a common factor.

$$\begin{aligned}
2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - 6x^{-\frac{1}{2}} &= 2x^{-\frac{1}{2}}(x^2 + 2x - 3) \\
&= 2x^{-\frac{1}{2}}(x - 1)(x + 3)
\end{aligned}$$

EXERCISE 1

1. Factor.

(a) $x^2 - x - 2$
(b) $x^2 - 9x + 14$
(c) $x^2 + 7x + 12$
(d) $2x^2 - x - 1$
(e) $5x^2 + 13x + 6$
(f) $6y^2 - 11y + 3$
(g) $t^3 + 2t^2 - 3t$
(h) $3x^4 + 7x^3 + 2x^2$

2. Factor.

(a) $4x^2 - 25$
(b) $x^3 - 1$
(c) $t^3 + 64$
(d) $y^3 - 9y$
(e) $8c^3 - 27d^3$
(f) $x^6 + 8$
(g) $x^4 - 16$
(h) $r^8 - 1$

3. Factor.

(a) $x^3 - x^2 - 16x + 16$
(b) $x^3 - 7x + 6$
(c) $x^3 + 5x^2 - 2x - 24$
(d) $x^3 + 2x^2 - 11x - 12$
(e) $4x^3 + 12x^2 + 5x - 6$
(f) $x^4 - 3x^3 - 7x^2 + 27x - 18$

4. Factor.

(a) $x^{\frac{5}{2}} - x^{\frac{1}{2}}$
(b) $x + 5 + 6x^{-1}$
(c) $x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 8x^{-\frac{1}{2}}$
(d) $2x^{\frac{7}{2}} - 2x^{\frac{1}{2}}$
(e) $1 + 2x^{-1} + x^{-2}$
(f) $(x^2 + 1)^{\frac{1}{2}} + 3(x^2 + 1)^{-\frac{1}{2}}$

Rationalizing

To rationalize a numerator or denominator that contains an expression such as

$$\sqrt{a} - \sqrt{b}$$

we multiply both the numerator and the denominator by the **conjugate radical**

$$\sqrt{a} + \sqrt{b}$$

Then we can take advantage of the formula for a difference of squares:

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$$

Example Rationalize the numerator in the expression

$$\frac{\sqrt{x+4} - 2}{x}$$

Solution We multiply the numerator and the denominator by the conjugate radical $\sqrt{x+4} + 2$:

$$\frac{\sqrt{x+4} - 2}{x} = \left(\frac{\sqrt{x+4} - 2}{x}\right)\left(\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}\right)$$

Do not expand the denominator.

$$= \frac{(x+4) - 4}{x(\sqrt{x+4} + 2)}$$

$$= \frac{x}{x(\sqrt{x+4} + 2)} \qquad (x \neq 0)$$

$$= \frac{1}{\sqrt{x+4} + 2}$$

EXERCISE 2

1. Rationalize the numerator.

(a) $\dfrac{\sqrt{x} - 3}{x - 9}$ (b) $\dfrac{\dfrac{1}{\sqrt{x}} - 1}{x - 1}$

(c) $\dfrac{x\sqrt{x} - 8}{x - 4}$ (d) $\dfrac{\sqrt{2 + h} + \sqrt{2 - h}}{h}$

(e) $\sqrt{x^2 + 3x + 4} - x$ (f) $\sqrt{x^2 + x} - \sqrt{x^2 - x}$

2. Rationalize the denominator.

(a) $\dfrac{1}{\sqrt{x + 1} - 1}$ (b) $\dfrac{4}{\sqrt{x + 2} + \sqrt{x}}$

(c) $\dfrac{x}{\sqrt{x^2 + 1} + x}$ (d) $\dfrac{x^2}{\sqrt{x + 1} - \sqrt{x - 1}}$

INTRODUCTION

In this first chapter, we show how the idea of a limit arises when we try to find a tangent to a curve. After developing the properties of limits of functions, we use them to compute tangents, velocities, and other rates of change. Then we investigate another type of limit, the limit of a sequence, and show how it is used to find the sum of an infinite series.

1.1 LINEAR FUNCTIONS AND THE TANGENT PROBLEM

A **linear function** is a function f of the form

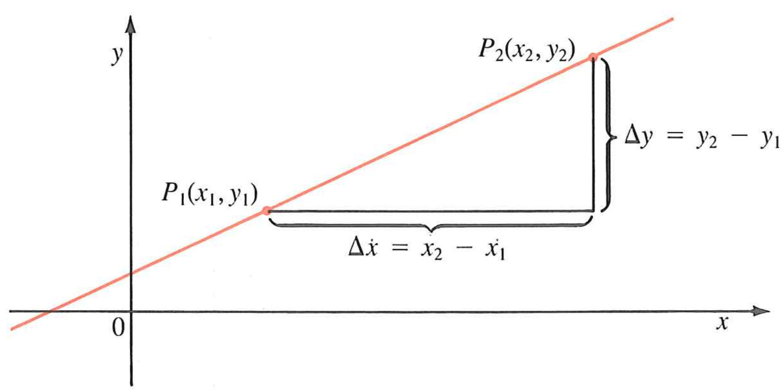
$$f(x) = mx + b, \quad m \text{ and } b \text{ constants}$$

It is called linear because its graph has the equation $y = mx + b$, which we recognize as the equation of a line with slope m and y-intercept b.

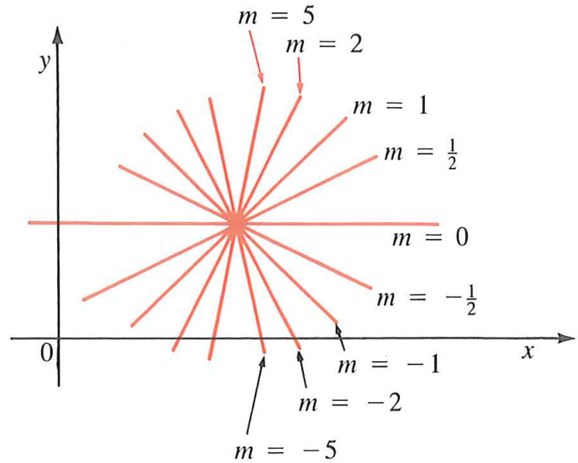
Recall that the **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Since the slope is the ratio of the change in y to the change in x, it can be interpreted as the **rate of change of y with respect to x.**



The following figure shows several lines labelled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones where the absolute value of the slope is the largest, and a horizontal line has slope zero. The slope of a vertical line is not defined.



Example 1 Find a linear function whose graph passes through the points $(-1, -1)$ and $(2, 5)$.

Solution The slope of the graph is

$$m = \frac{5-(-1)}{2-(-1)} = \frac{6}{3} = 2$$

We find the equation of the line using the point-slope form.

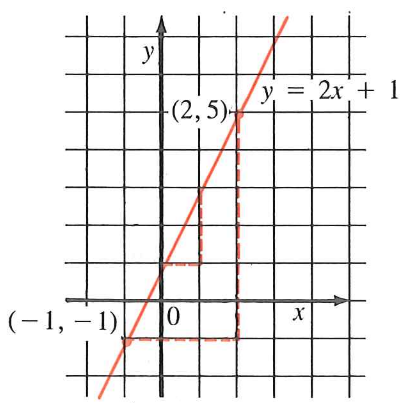
$$y - y_1 = m(x - x_1)$$
$$y - 5 = 2(x - 2)$$

or
$$y = 2x + 1$$

The function is given by

$$f(x) = 2x + 1$$

The fact that the rate of change of y with respect to x is 2 means that y increases twice as fast as x.



Example 2 A linear function is given by $y = 6 - 5x$. If x increases by 2, how does y change?

Solution The rate of change is

$$\frac{\Delta y}{\Delta x} = \text{slope} = -5$$

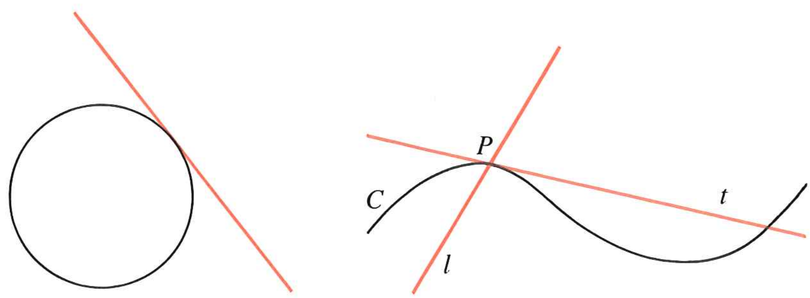
and we are given that $\Delta x = 2$. Thus

$$\Delta y = (-5)\Delta x = (-5)(2) = -10$$

and so y decreases by 10.

The Tangent Problem

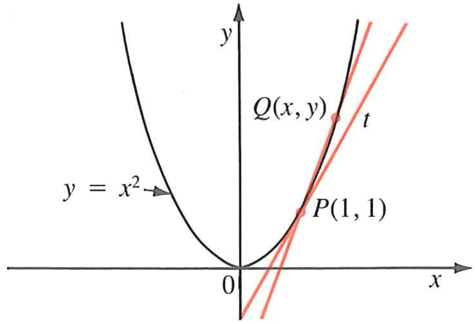
The word *tangent* comes from the Latin word *tangens*, which means touching. For a simple curve, such as a circle, a tangent is a line that intersects the circle once and only once. But for more complicated curves this definition is not good enough. The figure below shows a point P on a curve C and two lines l and t passing through P. The line l intersects C only once, but it does not look like a tangent. On the other hand, the line t looks like a tangent but it intersects C twice.



We look at the problem of finding a tangent line to a specific curve, $y = x^2$, in the following example.

Example 3 Find the equation of a tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution We will be able to find the equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But we can compute an approximation to m by choosing a nearby point $Q(x, y)$ on the parabola (as in the diagram) and computing the slope m_{PQ} of the line PQ, which is called a *secant line*.



We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{y - 1}{x - 1}$$

But since Q lies on the parabola we have $y = x^2$, so

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.1, 1.21)$ we have

$$m_{PQ} = \frac{1.21 - 1}{1.1 - 1} = \frac{0.21}{0.1} = 2.1$$

The following tables give the values of m_{PQ} for several values of x close to 1.

Approaching 1 From the Right

$x > 1$	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

Approaching 1 From the Left

$x < 1$	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The closer Q is to P, the closer x is to 1, and, it appears, the closer m_{PQ} is to 2.

This suggests that the slope of the tangent line t should be $m = 2$. We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m$$

and

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Since the tangent line passes through $P(1, 1)$ and has slope 2, we use the point-slope form of the equation of a line to write the equation of the tangent as

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

Example 3 shows that in order to solve tangent problems we must be able to find limits. After studying methods for computing limits in the next two sections we will return to the problem of finding tangent lines to general curves in Section 1.4.

EXERCISE 1.1

A **1.** State the slopes of the given linear functions.

(a) $y = 4x$ (b) $y = 3x - 5$

(c) $f(x) = \frac{1}{3}x - 2$ (d) $f(x) = 2 - 3x$

(e) $f(x) = \frac{1}{2}(1 - x)$ (f) $x + 2y = 3$

B **2.** Find an equation of the line that passes through the points $(-3, 5)$ and $(4, -5)$.

3. Find a linear function whose graph passes through the points $(-4, -2)$ and $(2, 10)$.

4. A linear function is given by $y = 16 + 3x$. How does y change

(a) if x increases by 4? (b) if x decreases by 2?

5. A linear function is given by $y = \dfrac{1 - x}{2}$. How does y change

(a) if x increases by 6? (b) if x decreases by 4?

6. A car travels at a constant speed and covers 140 km in 4 h. If s represents distance travelled (in kilometres) and t represents time elapsed (in hours), express s as a function of t and draw its graph. What does the slope of the line represent?

7. The point $P(1, 3)$ lies on the curve $y = 2x + x^2$.

(a) If Q is the point $(x, 2x + x^2)$, find the slope of the secant line PQ for the following values of x:

(i) 2 (ii) 1.5 (iii) 1.1 (iv) 1.01 (v) 1.001
(vi) 0 (vii) 0.5 (viii) 0.9 (ix) 0.99 (x) 0.999

(a) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(1, 3)$.

(c) Using the slope from part (b), find the equation of the tangent line to the curve at $P(1, 3)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

8. The point $P(2, 0)$ lies on the curve $y = -x^2 + 6x - 8$.

(a) If Q is the point $(x, -x^2+6x-8)$, find the slope of the secant line PQ for the following values of x:

(i) 3 (ii) 2.5 (iii) 2.1 (iv) 2.01
(v) 1 (vi) 1.5 (vii) 1.9 (viii) 1.99

(a) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, 0)$.

(c) Using the slope from part (b), find the equation of the tangent line to the curve at $P(2, 0)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

9. The point $P(1, \frac{1}{4})$ lies on the curve $y = \frac{1}{4}x^3$.

(a) If Q is the point $\left(x, \frac{1}{4}x^3\right)$, find the slope of the secant line PQ for the following values of x:

(i) 2 (ii) 1.5 (iii) 1.1 (iv) 1.01 (v) 1.001
(vi) 0 (vii) 0.5 (viii) 0.9 (ix) 0.99 (x) 0.999

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P\left(1, \frac{1}{4}\right)$.

(c) Using the slope from part (b), find the equation of the tangent line to the curve at $P\left(1, \frac{1}{4}\right)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

10. The point $P(0.5, 2)$ lies on the curve $y = \frac{1}{x}$.

(a) If Q is the point $\left(x, \frac{1}{x}\right)$, use your calculator to find approximate values of the slope of the secant line PQ for the following values of x:

(i) 2 (ii) 1 (iii) 0.9 (iv) 0.8 (v) 0.7
(vi) 0.6 (vii) 0.55 (viii) 0.51 (ix) 0.45 (x) 0.49

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 2)$.

(c) Using the slope from part (b), find the equation of the tangent line to the curve at $P(0.5, 2)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

C **11.** As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100 m rise, up to about 12 km.

(a) If the ground temperature is 20°C, find an expression for the temperature T as a function of the height h.

(b) Sketch the graph of T. What does the slope represent?

12. The monthly cost of owning a car depends on the number of kilometres driven. Judy Weyman found that in May it cost her \$500 to drive 800 km and in June it cost her \$650 to drive 1400 km.

(a) Express the monthly cost C as a function of distance driven d, assuming that a linear function is a suitable model.

(b) Use this function to predict the cost of driving 2000 km per month.
(c) What does the slope of the function represent?
(d) What is the monthly cost if she does not drive her car at all? Is it reasonable?
(e) Why is a linear function a suitable model in this situation?

1.2 THE LIMIT OF A FUNCTION

We saw in the first section how limits arise in trying to find a tangent line to a curve. Later in this chapter we will see that limits also arise in computing velocities and other rates of change. In fact, limits are basic to all of calculus and so in this section we look at limits in general and methods for calculating them.

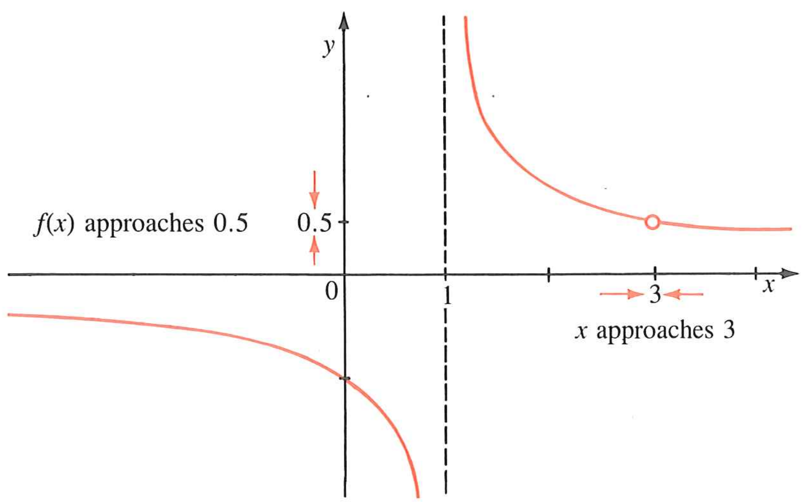
We begin by investigating the behaviour of the function

$$f(x) = \frac{x - 3}{x^2 - 4x + 3}$$

when x is near 3. The following table gives values of $f(x)$ for values of x approaching 3 (but not equal to 3).

$x < 3$	$f(x)$	$x > 3$	$f(x)$
2.5	0.666 667	3.5	0.400 000
2.9	0.526 316	3.1	0.476 190
2.99	0.502 513	3.01	0.497 512
2.999	0.500 250	3.001	0.499 750
2.9999	0.500 025	3.0001	0.499 975

The open circle at $\left(3, \frac{1}{2}\right)$ indicates that the function is not defined when $x = 3$.



From the table and the graph of f, we see that when x is close to 3 (on either side of 3), $f(x)$ is close to 0.5. In fact, it appears that we can make the values of $f(x)$ as close as we like to 0.5 by taking x close enough to 3. We express this by saying

$$\text{``the limit of } \frac{x-3}{x^2-4x+3} \text{ as } x \text{ approaches 3 is equal to } \tfrac{1}{2}\text{''}$$

and by writing

$$\lim_{x\to 3} \frac{x-3}{x^2-4x+3} = \frac{1}{2}$$

In general, we have the following definition of the limit of a function.

> We write $\quad \lim_{x\to a} f(x) = L$
>
> and say
>
> "the limit of $f(x)$, as x approaches a, equals L"
>
> if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a.

Roughly speaking, this says that the values of $f(x)$ become closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

Notice the phrase "but $x \neq a$" in the definition of a limit. This means that in finding the limit of $f(x)$ as x approaches a, we need never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. (The function f considered before the definition is not defined at $x = 3$.) The only thing that matters is how f is defined *near a*.

Example 1 Find $\lim_{x\to 5} (x^2 + 2x - 3)$.

Solution It seems clear that when x is close to 5, x^2 is close to 25 and $2x$ is close to 10. Thus it appears that

$$\lim_{x\to 5} (x^2 + 2x - 3) = 25 + 10 - 3 = 32$$

In Example 1 we arrived at the answer by intuitive reasoning, but it is also possible to find the limit using the following properties of limits. These properties are proved in more advanced courses in calculus using a precise definition of a limit. Notice that they apply only in situations where the limits exist. See Example 8 and Question 12 in Exercise 1.2 for examples in which limits do not exist.

Properties of Limits

Suppose that the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist and let c be a constant. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ if n is a positive integer
7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ if the root on the right side exists

These seven properties of limits can be stated verbally as follows.

1. The limit of a sum is the sum of the limits.
2. The limit of a difference is the difference of the limits.
3. The limit of a constant times a function is the constant times the limit of the function.
4. The limit of a product is the product of the limits.
5. The limit of a quotient is the quotient of the limits (if the limit of the denominator is not 0).
6. The limit of a power is the power of the limit.
7. The limit of a root is the root of the limit (if the root exists).

If we start with the basic limits

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c \qquad (c \text{ is a constant})$$

then from Properties 6 and 7 we deduce the following:

$$\lim_{x \to a} x^n = a^n \qquad \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \qquad (\text{if } \sqrt[n]{a} \text{ exists})$$

Using these limits, together with the seven properties of limits, we can compute limits of more complicated functions. First we return to the limit of Example 1.

Example 2 Find $\lim_{x\to 5} (x^2 + 2x - 3)$ using the properties of limits.

Solution

$$\begin{aligned}\lim_{x\to 5} (x^2 + 2x - 3) &= \lim_{x\to 5} x^2 + \lim_{x\to 5} 2x - \lim_{x\to 5} 3 &&\text{(Properties 1 and 2)}\\ &= \lim_{x\to 5} x^2 + 2\lim_{x\to 5} x - \lim_{x\to 5} 3 &&\text{(Property 3)}\\ &= 5^2 + 2(5) - 3\\ &= 32\end{aligned}$$

Example 3 Evaluate using the properties of limits.

(a) $\lim_{x\to 1} \dfrac{x^4 - 5x^2 + 1}{x + 2}$ (b) $\lim_{x\to 3} \sqrt{x^2 + x}$

Solution (a)

$$\begin{aligned}\lim_{x\to 1} \frac{x^4 - 5x^2 + 1}{x + 2} &= \frac{\lim_{x\to 1} (x^4 - 5x^2 + 1)}{\lim_{x\to 1} (x + 2)} &&\text{(Property 5)}\\ &= \frac{\lim_{x\to 1} x^4 - 5\lim_{x\to 1} x^2 + \lim_{x\to 1} 1}{\lim_{x\to 1} x + \lim_{x\to 1} 2} &&\text{(Properties 2, 3, and 1)}\\ &= \frac{1^4 - 5(1)^2 + 1}{1 + 2}\\ &= -1\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 3} \sqrt{x^2 + x} &= \sqrt{\lim_{x\to 3} (x^2 + x)} &&\text{(Property 7)}\\ &= \sqrt{\lim_{x\to 3} x^2 + \lim_{x\to 3} x} &&\text{(Property 1)}\\ &= \sqrt{3^2 + 3}\\ &= \sqrt{12}\\ &= 2\sqrt{3}\end{aligned}$$

Notice that if we let

$$f(x) = \frac{x^4 - 5x^2 + 1}{x + 2}$$

then

$$f(1) = \frac{1^4 - 5(1)^2 + 1}{1 + 2} = -1$$

and so we would have got the right answer in Example 3(a) by substituting 1 for x:

$$\lim_{x \to 1} f(x) = f(1)$$

Similarly, direct substitution provides the correct answer in Example 3(b):

$$\text{If } g(x) = \sqrt{x^2 + x}, \text{ then } \lim_{x \to 3} g(x) = g(3).$$

Functions with this property, that is,

$$\lim_{x \to a} f(x) = f(a)$$

are called **continuous at *a***. The geometric properties of such functions will be studied in the next section.

Using the properties of limits, it can be shown that many familiar functions are continuous. Recall that a **polynomial** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are constants. A **rational function** is a ratio of polynomials.

(a) Any polynomial P is continuous at every number; that is,

$$\lim_{x \to a} P(x) = P(a)$$

(b) Any rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials, is continuous at every number a such that $Q(a) \neq 0$; that is,

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} \qquad Q(a) \neq 0$$

For instance, we could rework the solution to Example 2 as follows:

$f(x) = x^2 + 2x - 3$ is a polynomial, so it is continuous at 5

and therefore,

$$\lim_{x \to 5} (x^2 + 2x - 3) = f(5) = 5^2 + 2(5) - 3 = 32$$

Not all limits can be evaluated by direct substitution, however, as the following examples illustrate.

Example 4 Evaluate $\lim_{x \to 4} \dfrac{x^2 - 16}{x - 4}$.

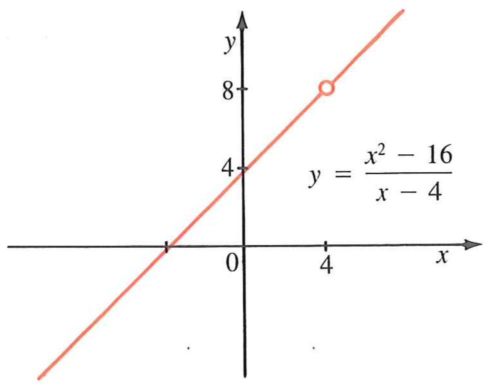
Solution Let

$$f(x) = \frac{x^2 - 16}{x - 4}$$

We cannot find the limit by substituting $x = 4$ because $f(4)$ is not defined ($\frac{0}{0}$ is meaningless). Remember that the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but not equal to a. Therefore in this example we have $x \neq 4$, so we can factor the numerator as a difference of squares and write

$$\begin{aligned}\lim_{x \to 4} \frac{x^2 - 16}{x - 4} &= \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} \\ &= \lim_{x \to 4} (x + 4) \\ &= 4 + 4 \\ &= 8\end{aligned}$$

Notice that in Example 4 we replaced the given rational function by a continuous function $[g(x) = x + 4]$ that is equal to $f(x)$ for $x \neq 4$. This is illustrated by the graph of f.



Example 5 Find $\lim_{x \to 2} \dfrac{x^3 - 8}{x^2 - 3x + 2}$.

Solution Notice that we cannot substitute $x = 2$ since we would obtain $\frac{0}{0}$. We replace the given rational function by a rational function that is continuous at 2. To do this, we factor the numerator by using the formula for a difference of cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

with $a = x$ and $b = 2$. Then

$$\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 3x + 2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x-1)}$$
$$= \lim_{x\to 2} \frac{x^2+2x+4}{x-1}$$
$$= \frac{2^2 + 2(2) + 4}{2-1}$$
$$= 12$$

Example 6 Find $\lim_{h\to 0} \frac{(2+h)^2 - 4}{h}$.

Solution Again we cannot compute the limit by letting $h = 0$, so we first simplify the numerator:

$$\lim_{h\to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h\to 0} \frac{(4 + 4h + h^2) - 4}{h}$$
$$= \lim_{h\to 0} \frac{4h + h^2}{h}$$
$$= \lim_{h\to 0} (4 + h)$$
$$= 4$$

Example 7 Evaluate $$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x}.$$

Solution Here the algebraic simplification consists of rationalizing the numerator, that is, multiplying numerator and denominator by the conjugate radical $\sqrt{x+1} + 1$:

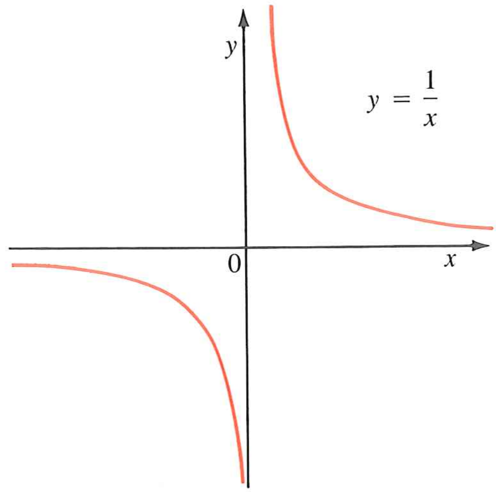
Do not expand the denominator.

$$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x\to 0} \left(\frac{\sqrt{x+1} - 1}{x}\right)\left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}\right)$$
$$= \lim_{x\to 0} \frac{(x+1) - 1}{x(\sqrt{x+1} + 1)}$$
$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+1} + 1)}$$
$$= \lim_{x\to 0} \frac{1}{\sqrt{x+1} + 1}$$
$$= \frac{\lim_{x\to 0} 1}{\sqrt{\lim_{x\to 0} (x+1)} + \lim_{x\to 0} 1}$$
$$= \frac{1}{\sqrt{0+1} + 1}$$
$$= \frac{1}{2}$$

Example 8 Show that $\lim_{x \to 0} \frac{1}{x}$ does not exist.

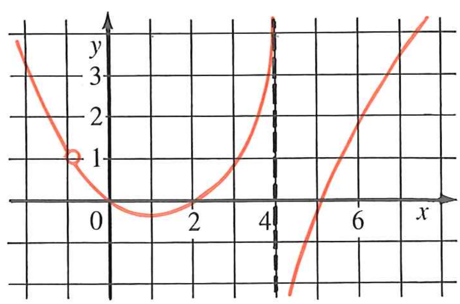
Solution As x approaches 0 through positive values, $\frac{1}{x}$ becomes very large. As x approaches 0 through negative values, $\frac{1}{x}$ becomes very large negative. We see from the graph of $y = \frac{1}{x}$ that the values of y do not approach any number as x approaches 0. Therefore

$$\lim_{x \to 0} \frac{1}{x} \text{ does not exist}$$



EXERCISE 1.2

A **1.** Use the given graph of f to state the value of the limit, if it exists.



(a) $\lim_{x \to 3} f(x)$ (b) $\lim_{x \to 2} f(x)$ (c) $\lim_{x \to -1} f(x)$ (d) $\lim_{x \to 4} f(x)$

2. State the value of each limit.

(a) $\lim_{x\to 2} x^3$ (b) $\lim_{x\to \pi} x$

(c) $\lim_{x\to 8} 3$ (d) $\lim_{x\to 4} \sqrt{x}$

(e) $\lim_{x\to k} x^6$ (f) $\lim_{x\to 0} \pi$

B **3.** Use the properties of limits to evaluate the following.

(a) $\lim_{x\to 1} (3x - 7)$ (b) $\lim_{x\to -1} (2x^2 - 5x + 3)$

(c) $\lim_{x\to 2} (x^3 + x^2 - 2x - 8)$ (d) $\lim_{x\to -2} (x^2 + 5x + 3)^6$

(e) $\lim_{x\to 0} \dfrac{x - 1}{x + 1}$ (f) $\lim_{x\to 4} \dfrac{x^2 + 2x - 3}{x^2 + 2}$

(g) $\lim_{t\to 2} \dfrac{t^4 - 3t + 1}{t^2(t - 1)^3}$ (h) $\lim_{u\to -4} \sqrt{u^4 + 2u^2}$

(i) $\lim_{x\to 5} \sqrt[3]{x^2 + 2x - 8}$ (j) $\lim_{t\to 3} \left(2t^2 + \sqrt{\dfrac{6 + t}{4 - t}}\right)$

4. Find the following limits.

(a) $\lim_{x\to -2} \dfrac{x + 2}{x^2 - 4}$ (b) $\lim_{x\to 1} \dfrac{x^2 - 3x + 2}{x - 1}$

(c) $\lim_{x\to 3} \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3}$ (d) $\lim_{x\to -2} \dfrac{2x^2 + 5x + 2}{x^2 - 2x - 8}$

(e) $\lim_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1}$ (f) $\lim_{x\to -3} \dfrac{x + 3}{x^3 + 27}$

(g) $\lim_{x\to 9} \dfrac{x - 9}{\sqrt{x} - 3}$ (h) $\lim_{x\to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

5. Evaluate the following.

(a) $\lim_{h\to 0} \dfrac{(4 + h)^3 - 64}{h}$ (b) $\lim_{h\to 0} \dfrac{(h - 2)^2 - 4}{h}$

(c) $\lim_{h\to 0} \dfrac{\frac{1}{1 + h} - 1}{h}$ (d) $\lim_{h\to 0} \dfrac{(2 + h)^4 - 16}{h}$

(e) $\lim_{h\to 0} \dfrac{\sqrt{9 + h} - 3}{h}$ (f) $\lim_{h\to 0} \dfrac{\frac{1}{(2 + h)^2} - \frac{1}{4}}{h}$

6. Find the following limits, if they exist.

(a) $\lim_{x\to 3} \dfrac{1}{(x - 3)^2}$ (b) $\lim_{x\to -8} \dfrac{x^2 + 16x + 64}{x + 8}$

(c) $\lim_{x\to 1} \dfrac{x^4 - 1}{x - 1}$ (d) $\lim_{x\to -1} \dfrac{x - 1}{x^2 - 1}$

(e) $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - 2x + 1}$ (f) $\lim_{x\to -2} \dfrac{x^2 - x - 2}{x^2 + 3x + 2}$

(g) $\lim_{x\to 3} \dfrac{x^{-2} - 3^{-2}}{x - 3}$

(h) $\lim_{x\to 4} \dfrac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{2}}{x - 4}$

(i) $\lim_{x\to 1} \dfrac{x^3 - 1}{x^3 - x^2 - 4x + 4}$

(j) $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x} - x}$

7. (a) Use your calculator to evaluate $f(x) = (1 + x)^{\frac{1}{x}}$ correct to six decimal places for $x = 1, 0.1, 0.01, 0.001, 0.000\,1, 0.000\,01, 0.000\,001$, and $0.000\,000\,1$.

(b) Estimate the value of the limit

$$\lim_{x\to 0} (1 + x)^{\frac{1}{x}}$$

to five decimal places.

8. (a) Use your calculator to evaluate $g(x) = \dfrac{2^x - 1}{x}$ correct to four decimal places for $x = 1, 0.1, 0.01, 0.001, 0.000\,1$.

(b) Estimate the value of the limit

$$\lim_{h\to 0} \frac{2^h - 1}{h}$$

to three decimal places.

C **9.** Evaluate the following limits.

(a) $\lim_{x\to 8} \dfrac{x - 8}{\sqrt[3]{x} - 2}$

(b) $\lim_{x\to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$

10. If $f(x) = 2x + 3$, show that

$$|f(x) - 7| < 0.01 \quad \text{if} \quad |x - 2| < 0.005$$

11. How close to 1 do we have to take x so that $\dfrac{16x^2 - 1}{4x - 1}$ is within a distance of 0.001 from 5?

12. Show that $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist.

13. Find functions f and g such that $\lim_{x\to 0} [f(x) + g(x)]$ exists but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. [Hint: See Example 8.]

PROBLEMS PLUS

1. Evaluate $\lim_{x\to 1} \dfrac{\sqrt{x} - 1}{\sqrt[3]{x} - 1}$. *Hint*: Introduce a new variable $t = \sqrt[6]{x}$.

1.3 ONE-SIDED LIMITS

The functions we have considered so far have been defined by simple formulas, but there are many functions that cannot be described in this way. Here are some examples: The population of Ottawa as a function of time; the cost of a taxi ride as a function of distance; the cost of mailing a first-class letter as a function of its mass. Such functions can be given by different formulas in different parts of their domains.

Consider the function f described by

$$f(x) = \begin{cases} x^2 & \text{if } x \leqslant 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

Remember that a function is a rule. For this particular function the rule is the following: First look at the value of x. If it happens that $x \leqslant 1$, then the value of $f(x)$ is x^2. On the other hand, if $x > 1$, then the value of $f(x)$ is $3 - x$. For instance, we compute $f(0)$, $f(1)$, and $f(2)$ as follows:

Since $0 \leqslant 1$, we have $f(0) = 0^2 = 0$.
Since $1 \leqslant 1$, we have $f(1) = 1^2 = 1$.
Since $2 > 1$, we have $f(2) = 3 - 2 = 1$.

We now investigate the limiting behaviour of $f(x)$ as x approaches 1.

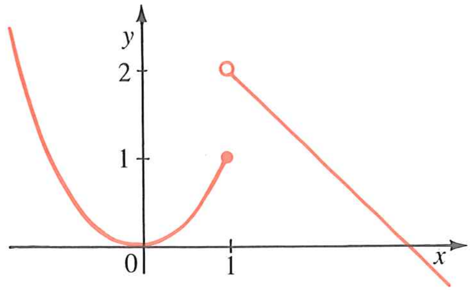
Approaching From the Left		Approaching From the Right	
$x < 1$	$f(x) = x^2$	$x > 1$	$f(x) = 3 - x$
0.9	0.81	1.1	1.9
0.99	0.980 1	1.01	1.99
0.999	0.998 001	1.001	1.999

We see from the tables that $f(x)$ approaches 1 as x approaches 1 from the left, but $f(x)$ approaches 2 as x approaches 1 from the right. The notation we use to indicate this is

$$\lim_{x \to 1^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 2$$

Notice that the ordinary two-sided limit $\lim_{x \to 1} f(x)$ does not exist because the function approaches different values from the left and right.

Further insight into this type of function is gained from its graph. We observe that if $x \leqslant 1$, then $f(x) = x^2$, so the part of the graph of f that lies to the left of $x = 1$ must coincide with the graph of the parabola $y = x^2$. If $x > 1$, then $f(x) = 3 - x$, so the part of the graph of f that lies to the right of $x = 1$ coincides with the graph of $y = 3 - x$, which is a line with slope -1. The solid circle indicates that the point is included on the graph; the open circle indicates that the point is excluded from the graph.



In general, we write

$$\lim_{x \to a^-} f(x) = L$$

and say

"the **left-hand limit** of $f(x)$, as x approaches a, equals L"

or "the **limit of $f(x)$ as x approaches a from the left** equals L"

if the values of $f(x)$ can be made close to L by taking x close to a with $x < a$.

Similarly, if we consider only $x > a$, we have the **right-hand limit**:

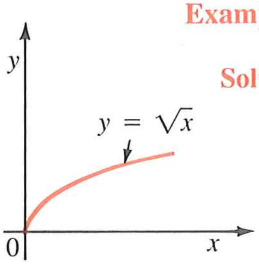
$$\lim_{x \to a^+} f(x) = L$$

If a function has different expressions to the left and right of the number a, the following theorem provides a convenient way to test whether or not $\lim_{x \to a} f(x)$ exists.

> If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.
>
> If $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x) = L$.

When computing one-sided limits, we use the fact that the properties of limits listed in Section 1.2 also hold for one-sided limits.

Example 1 Find $\lim_{x \to 0^+} \sqrt{x}$.

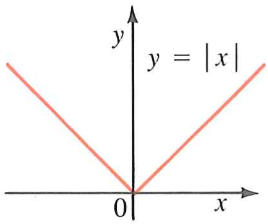


Solution Notice that the function $f(x) = \sqrt{x}$ is defined only for $x \geqslant 0$, so the two-sided limit $\lim_{x \to 0} \sqrt{x}$ does not make sense. If we let x approach 0 while restricting x to be positive, we see that $\sqrt{x}$ approaches 0:

$$\lim_{x \to 0^+} \sqrt{x} = \sqrt{\lim_{x \to 0^+} x} = \sqrt{0} = 0$$

Example 2 Show that $\lim_{x \to 0} |x| = 0$.

Solution Recall that



$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Therefore $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

and $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$

Since the left and right limits are equal, we have

$$\lim_{x \to 0} |x| = 0$$

Example 3 The **Heaviside function** H is defined by

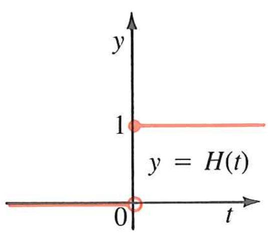
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$. Evaluate, if possible,

(a) $\lim_{t \to 0^-} H(t)$ (b) $\lim_{t \to 0^+} H(t)$ (c) $\lim_{t \to 0} H(t)$

Solution (a) Since $H(t) = 0$ for $t < 0$, we have

$$\lim_{t \to 0^-} H(t) = \lim_{t \to 0^-} 0 = 0$$



(b) Since $H(t) = 1$ for $t > 0$, we have

$$\lim_{t \to 0^+} H(t) = \lim_{t \to 0^+} 1 = 1$$

(c) We see that $\lim_{t \to 0^-} H(t) \neq \lim_{t \to 0^+} H(t)$ and so $\lim_{t \to 0} H(t)$ does not exist.

Example 4 If

$$f(x) = \begin{cases} -x - 2 & \text{if } x \leqslant -1 \\ x & \text{if } -1 < x < 1 \\ x^2 - 2x & \text{if } x \geqslant 1 \end{cases}$$

determine whether or not $\lim_{x \to -1} f(x)$ and $\lim_{x \to 1} f(x)$ exist.

Solution We first compute the one-sided limits. Since $f(x) = -x - 2$ for $x < -1$, we have

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-x - 2) = -(-1) - 2 = -1$$

Since $f(x) = x$ for $-1 < x < 1$, we have

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1$$

The left and right limits are equal, so

$$\lim_{x \to -1} f(x) = -1$$

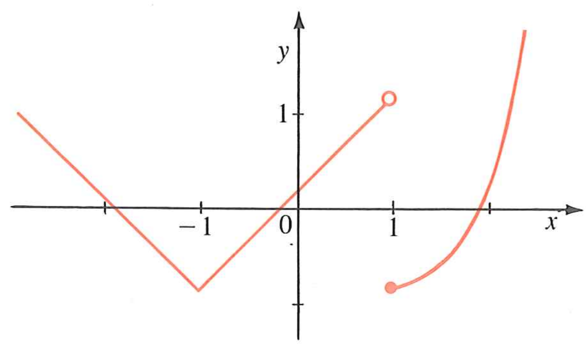
Similarly, we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 2x) = 1^2 - 2(1) = -1$$

The left and right limits are different, so

$$\lim_{x \to 1} f(x) \text{ does not exist}$$

This information is shown in the graph of f.



Discontinuities

We recall from Section 1.2 the definition of a continuous function.

> f is **continuous** at a number a if
> $$\lim_{x \to a} f(x) = f(a)$$

Implicitly, this requires three things if f is continuous at a.

1. $f(a)$ is defined (so a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If f is not continuous at a, we say f is **discontinuous** at a, or f has a **discontinuity** at a.

For instance, the Heaviside function in Example 3 has a discontinuity at $t = 0$ because $\lim_{t \to 0} H(t)$ does not exist. Notice that there is a break in the graph of H at $t = 0$. This is typical of functions that have discontinuities. In fact, you can think of a continuous function as a function whose graph has no holes or breaks. You can draw its graph without removing your pencil from the paper. Discontinuities occur where there are breaks in the graph.

Example 5 Where are the following functions discontinuous?

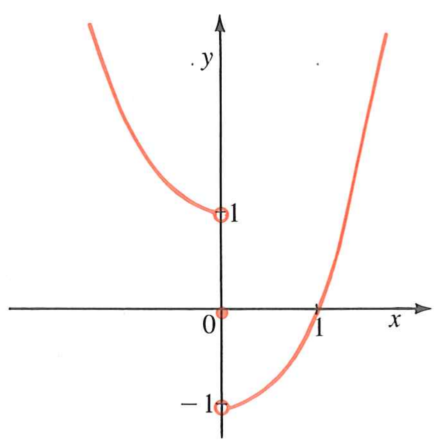
(a) $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^2 - 1 & \text{if } x > 0 \end{cases}$ (b) $g(x) = \begin{cases} x + 1 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Solution (a) When $x < 0$, we have $f(x) = x^2 + 1$, and we know polynomials are continuous. Similarly, $f(x) = x^2 - 1$ for $x > 0$. So f is continuous when $x \neq 0$. The only possibility for a discontinuity is $x = 0$, so we try to compute $\lim_{x \to 0} f(x)$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 1) = 0^2 + 1 = 1$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 - 1) = 0^2 - 1 = -1$$

Since the left and right limits are different, $\lim_{x \to 0} f(x)$ does not exist.

Therefore f is discontinuous at 0. This can also be seen from the break in the graph of f.



(b) The only possibility for a discontinuity is $x = 2$. Since $g(x) = x + 1$ for $x \neq 2$, we have

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} (x + 1) = 2 + 1 = 3$$

But, by definition, $g(2) = 1$

So $\lim_{x \to 2} g(x) \neq g(2)$

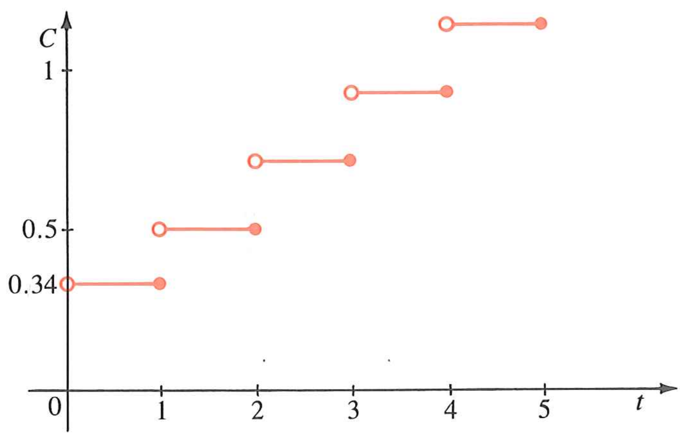
Therefore g is discontinuous at 2.

Example 6 The cost of a long-distance night-time phone call from Pine Bay to Hester is 26¢ for the first minute and 22¢ for each additional minute (or part of a minute). There is a minimum charge of 34¢ on all calls. Draw the graph of the cost C (in dollars) of a phone call as a function of the time t (in minutes). Where are the discontinuities of this function?

Solution From the given information, we have

$$\begin{aligned} C(t) &= 0.34 && \text{if } 0 < t \leqslant 1 \\ C(t) &= 0.26 + 0.22 = 0.48 && \text{if } 1 < t \leqslant 2 \\ C(t) &= 0.26 + 2(0.22) = 0.70 && \text{if } 2 < t \leqslant 3 \\ C(t) &= 0.26 + 3(0.22) = 0.92 && \text{if } 3 < t \leqslant 4 \end{aligned}$$

and so on.



From the graph we see that there are discontinuities when $t = 1, 2, 3, \ldots$. For instance, the discontinuity at $t = 2$ occurs because

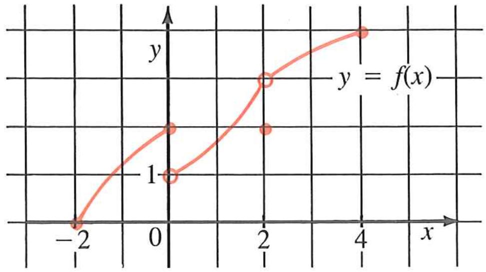
$$\lim_{t \to 2^-} C(t) = 0.48 \quad \text{and} \quad \lim_{t \to 2^+} C(t) = 0.70$$

and so $\lim_{t \to 2} C(t)$ does not exist.

The function in Example 6 is called a **step function** because of the appearance of its graph.

EXERCISE 1.3

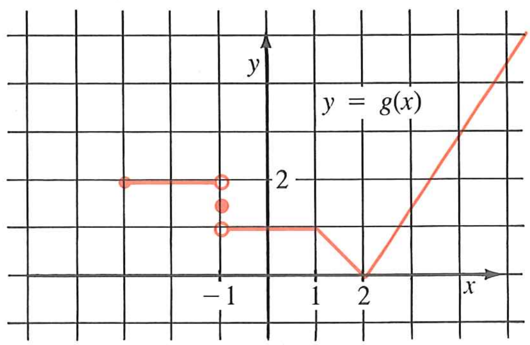
A **1.** Use the given graph of f to state the value of the limit, if it exists.



(a) $\lim_{x \to -2^+} f(x)$ (b) $\lim_{x \to 0^-} f(x)$ (c) $\lim_{x \to 0^+} f(x)$ (d) $\lim_{x \to 0} f(x)$

(e) $\lim_{x \to 2^-} f(x)$ (f) $\lim_{x \to 2^+} f(x)$ (g) $\lim_{x \to 2} f(x)$ (h) $\lim_{x \to 4^-} f(x)$

2. Use the given graph of g to state the value of the limit, if it exists.

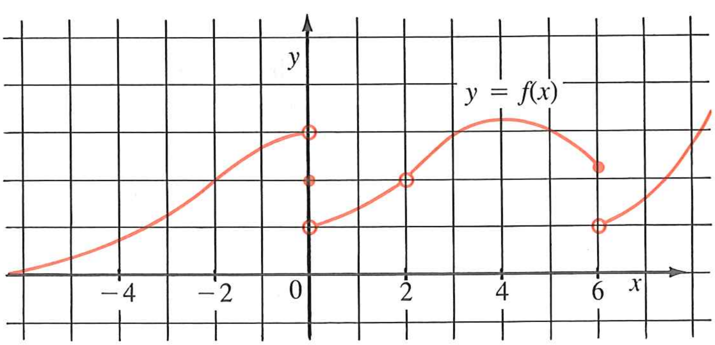


(a) $\lim_{x \to -3^+} g(x)$ (b) $\lim_{x \to -1^-} g(x)$ (c) $\lim_{x \to -1^+} g(x)$ (d) $\lim_{x \to -1} g(x)$

(e) $\lim_{x \to 2^-} g(x)$ (f) $\lim_{x \to 2^+} g(x)$ (g) $\lim_{x \to 2} g(x)$ (h) $\lim_{x \to 1} g(x)$

3. The graph of f is given. State whether f is continuous or discontinuous at each of the following numbers.

(a) -2 (b) 0 (c) 2 (d) 4 (e) 6



B **4.** Find the following limits, if they exist.

(a) $\lim_{x \to 0^+} \sqrt[4]{x}$ (b) $\lim_{x \to 3^+} \sqrt{x - 3}$

(c) $\lim_{x \to 1^-} \sqrt{1 - x}$ (d) $\lim_{x \to \frac{1}{2}^-} \sqrt[4]{1 - 2x}$

(e) $\lim_{x \to 6^+} |x - 6|$ (f) $\lim_{x \to 6^-} |x - 6|$

(g) $\lim_{x \to 6} |x - 6|$ (h) $\lim_{x \to 0^+} \frac{|x|}{x}$

(i) $\lim_{x \to 0^-} \frac{|x|}{x}$ (j) $\lim_{x \to 0} \frac{|x|}{x}$

5. Let

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ x + 1 & \text{if } x \geqslant 0 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of f.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

6. Let

$$g(x) = \begin{cases} x^2 & \text{if } x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of g.

(a) $\lim_{x \to 1^-} g(x)$ (b) $\lim_{x \to 1^+} g(x)$ (c) $\lim_{x \to 1} g(x)$

7. Let

$$h(x) = \begin{cases} 1 - x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -x - 1 & \text{if } x > 0 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of h.

(a) $\lim_{x \to 0^-} h(x)$ (b) $\lim_{x \to 0^+} h(x)$ (c) $\lim_{x \to 0} h(x)$

8. Let

$$f(x) = \begin{cases} -1 & \text{if } x \leqslant -2 \\ \frac{1}{2}x & \text{if } -2 < x < 2 \\ 1 & \text{if } x \geqslant 2 \end{cases}$$

(a) Find the following limits.

(i) $\lim_{x \to -2^-} f(x)$ (ii) $\lim_{x \to -2^+} f(x)$

(iii) $\lim_{x \to 2^-} f(x)$ (iv) $\lim_{x \to 2^+} f(x)$

(b) Sketch the graph of f.

(c) Where is f discontinuous?

9. Let

$$f(x) = \begin{cases} (x + 1)^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x \leqslant 1 \\ 2x - x^2 & \text{if } x > 1 \end{cases}$$

(a) Find the following limits, if they exist.

(i) $\lim_{x \to -1^-} f(x)$ (ii) $\lim_{x \to -1^+} f(x)$ (iii) $\lim_{x \to -1} f(x)$

(iv) $\lim_{x \to 1^-} f(x)$ (v) $\lim_{x \to 1^+} f(x)$ (vi) $\lim_{x \to 1} f(x)$

(b) Sketch the graph of f.

(c) Where is f discontinuous?

10. Where are the following functions discontinuous?

(a) $f(x) = \begin{cases} 2x + 3 & \text{if } x \neq 4 \\ 12 & \text{if } x = 4 \end{cases}$

(b) $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ x + 1 & \text{if } 0 < x \leqslant 1 \\ (x - 1)^2 & \text{if } x > 1 \end{cases}$

(c) $f(x) = \begin{cases} -x & \text{if } x < -1 \\ x^3 & \text{if } -1 \leqslant x \leqslant 1 \\ x & \text{if } x > 1 \end{cases}$

(d) $f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ x - 2 & 1 < x < 3 \\ x - 4 & \text{if } 3 \leqslant x \leqslant 4 \end{cases}$

11. Postal rates for a first-class letter up to 200 g are given in the following chart.

Up to and including	30 g	50 g	100 g	200 g
Mailing cost	\$0.38	0.59	0.76	1.14

Draw the graph of the cost C (in dollars) of mailing a first-class letter as a function of its mass x (in grams). Where are the discontinuities of this function?

12. A taxi company charges \$1.00 for the first 0.2 km (or part) and \$0.10 for each additional 0.1 km (or part). Draw the graph of the cost C of a taxi ride, in dollars, as a function of the distance travelled x (in kilometres). Where are the discontinuities of this function?

C **13.** Let

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leqslant 1 \\ |x| - 1 & \text{if } 1 < |x| \leqslant 2 \\ (x - 3)^2 & \text{if } x > 2 \\ (x + 3)^2 & \text{if } x < -2 \end{cases}$$

Sketch the graph of f and determine any values of x at which f is discontinuous.

14. For what value of the constant c is the function

$$f(x) = \begin{cases} x + c & \text{if } x < 2 \\ cx^2 + 1 & \text{if } x \geqslant 2 \end{cases}$$

continuous at every number?

PROBLEMS PLUS

The *greatest integer function* is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. For instance, $[\![6]\!] = 6$, $[\![6.83]\!] = 6$, $[\![\pi]\!] = 3$, and $[\![-4.2]\!] = -5$.

(a) Sketch the graph of this function.

(b) Find $\lim_{x\to 3^-} [\![x]\!]$ and $\lim_{x\to 3^+} [\![x]\!]$.

(c) For what values of a does $\lim_{x\to a} [\![x]\!]$ exist?

(d) For what values of a is the greatest integer function discontinuous?

(e) Sketch the graph of $g(x) = [\![2x + 1]\!]$. Where is it discontinuous?

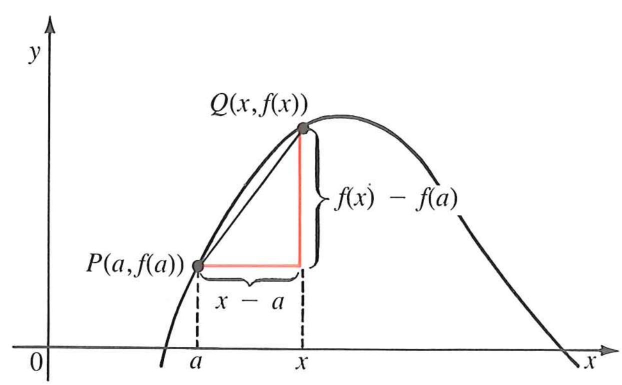
(f) Sketch the graph of $h(x) = x - [\![x]\!]$.

1.4 USING LIMITS TO FIND TANGENTS

In Section 1.1 we found the tangent line to the parabola $y = x^2$ at the point (1, 1) by computing its slope as the limit of slopes of secant lines.

In general, if a curve C has equation $y = f(x)$ and we want to find the tangent to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

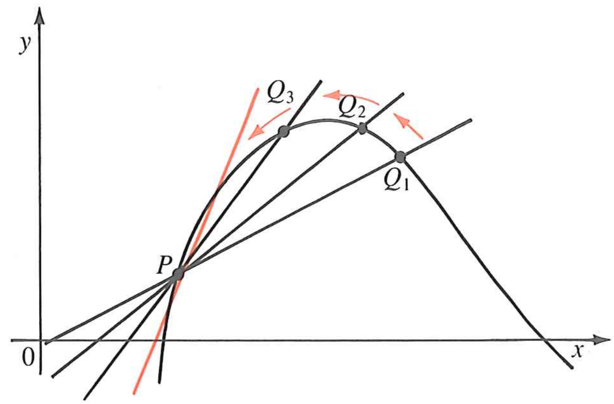
$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$



Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the **tangent** to be the line through P with slope m. In the notation of limits we write

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad (1)$$

This definition of the tangent amounts to saying that the tangent line is the limiting position of the sequence of secant lines PQ_1, PQ_2, PQ_3, ... in the figure below as the points Q_1, Q_2, Q_3, ... approach P along the curve.



Example 1 (a) Find the slope and the equation of the tangent line to the curve $y = 2x^2 + 4x - 1$ at the point $(2, 15)$.
(b) Sketch the curve and the tangent line.

Solution (a) We find the slope of the tangent line by using Formula 1 with $a = 2$ and $f(x) = 2x^2 + 4x - 1$:

$$\begin{aligned} m &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \to 2} \frac{(2x^2 + 4x - 1) - [2(2)^2 + 4(2) - 1]}{x - 2} \\ &= \lim_{x \to 2} \frac{2x^2 + 4x - 16}{x - 2} \\ &= \lim_{x \to 2} \frac{2(x^2 + 2x - 8)}{x - 2} \\ &= \lim_{x \to 2} \frac{2(x - 2)(x + 4)}{x - 2} \\ &= \lim_{x \to 2} 2(x + 4) \\ &= 2(2 + 4) \\ &= 12 \end{aligned}$$

$y - y_1 = m(x - x_1)$

The slope of the tangent line at (2, 15) is 12. Using the point-slope form, we find that the equation of the tangent line is

$$y - 15 = 12(x - 2)$$

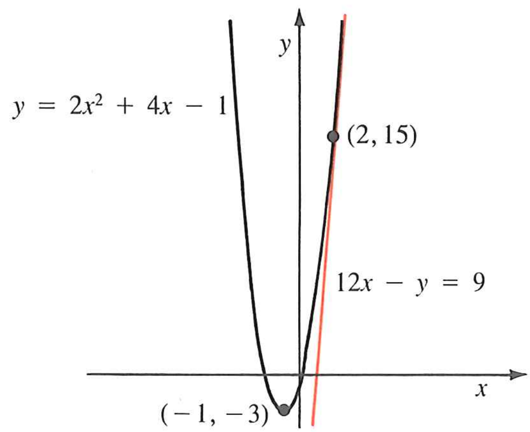
which simplifies to

$$12x - y - 9 = 0$$

(b) Recall that to graph a quadratic function we complete the square.

$$\begin{aligned} y &= 2x^2 + 4x - 1 \\ &= 2(x^2 + 2x) - 1 \\ &= 2[(x + 1)^2 - 1] - 1 \\ &= 2(x + 1)^2 - 3 \end{aligned}$$

The graph is a parabola with vertex $(-1, -3)$ that opens upward.

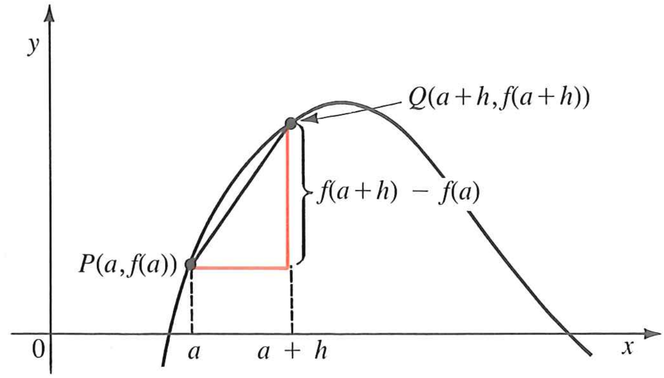


Another expression for the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

(The diagram illustrates the case where $h > 0$ and Q is to the right of P. If $h < 0$, however, Q would be to the left of P.) Notice that as x approaches a, h approaches 0, and so the expression for the slope of the tangent line becomes

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \qquad (2)$$



Example 2 Find the tangent line to the hyperbola $xy = 1$ at the point $\left(-2, -\frac{1}{2}\right)$.

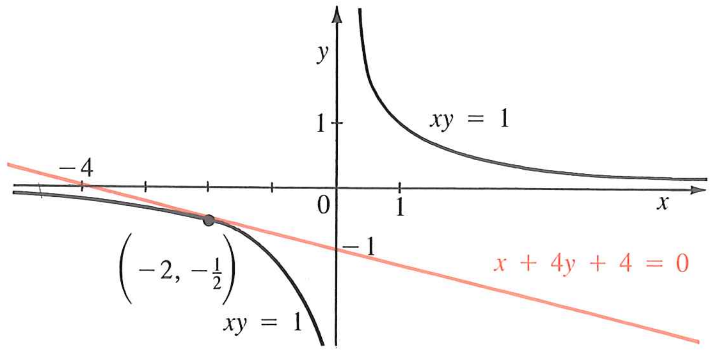
Solution The equation of the hyperbola can be written as $y = \frac{1}{x}$. Thus, using Formula 2 with $f(x) = \frac{1}{x}$, we obtain the slope of the tangent line:

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{-2+h} - \dfrac{1}{-2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{-2+h} + \dfrac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{2+(-2+h)}{2(-2+h)}}{h} \\
&= \lim_{h\to 0} \frac{h}{2(-2+h)h} \\
&= \lim_{h\to 0} \frac{1}{2(-2+h)} \\
&= \frac{1}{2(-2)} \\
&= -\frac{1}{4}
\end{aligned}
$$

The equation of the tangent line at $\left(-2, -\frac{1}{2}\right)$ is

$$
\begin{aligned}
y + \tfrac{1}{2} &= -\tfrac{1}{4}(x+2) \\
4y + 2 &= -x - 2 \\
x + 4y + 4 &= 0
\end{aligned}
$$

The hyperbola and the tangent line are shown in the diagram.



Example 3 (a) Find the tangent line to the curve $y = \sqrt{x-2}$ at the point $(6, 2)$.

(b) Graph the curve and the tangent line.

Solution (a) We find the slope using Formula 2 with $f(x) = \sqrt{x-2}$.

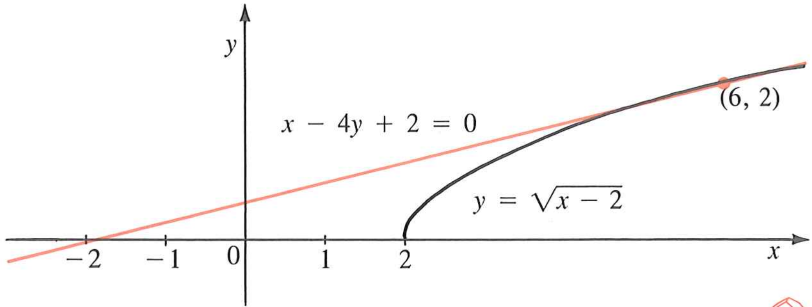
Rationalize the numerator.

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(6+h) - f(6)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{(6+h)-2} - \sqrt{6-2}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) \\ &= \lim_{h\to 0} \frac{(4+h) - 4}{h(\sqrt{4+h}+2)} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} \\ &= \frac{1}{\sqrt{4+0}+2} \\ &= \frac{1}{4} \end{aligned}$$

The equation of the tangent line at $(6, 2)$ is

$$\begin{aligned} y - 2 &= \tfrac{1}{4}(x - 6) \\ 4y - 8 &= x - 6 \\ x - 4y + 2 &= 0 \end{aligned}$$

(b) We graph the curve $y = \sqrt{x-2}$ by taking the square root function $y = \sqrt{x}$ and shifting it two units to the right.



EXERCISE 1.4

B **1.** (a) Find the slope of the tangent line to the parabola $y = 2x - x^2$ at the point $(2, 0)$.
(i) using Formula 1 (ii) using Formula 2
(b) Find the equation of the tangent line.
(c) Graph the parabola and the tangent line.

2. (a) Find the slope of the tangent line to the cubic curve $y = x^3$ at the point $(1, 1)$.
(i) using Formula 1 (ii) using Formula 2
(b) Find the equation of the tangent line.
(c) Graph the curve and the tangent line.

3. Find the slope in Example 1 using Formula 2.

4. Find the slope in Example 2 using Formula 1.

5. Find the slope in Example 3 using Formula 1.

6. (a) Find the slope of the tangent lines to the parabola $y = x^2 + 4x - 1$ at the points whose x-coordinates are given.
(i) -3 (ii) -2 (iii) 0
(b) Graph the parabola and the three tangents.

7. For each of the following curves
(a) find the slope of the tangent at the given point,
(b) find an equation of the tangent at the given point,
(c) graph the curve and the tangent.
(i) $y = 4 - x^2$ at $(-2, 0)$
(ii) $y = x^2 - 6x + 5$ at $(2, -3)$
(iii) $y = 1 - x^3$ at $(0, 1)$
(iv) $y = \dfrac{1}{x-1}$ at $\left(3, \frac{1}{2}\right)$
(v) $y = \sqrt{x+3}$ at $(6, 3)$
(vi) $y = 2x^4$ at $(-1, 2)$

8. Find the equation of the tangent line to the graph of the given function at the given point.

(a) $f(x) = 4 - x + 3x^2$, $(-1, 8)$

(b) $f(x) = x^3 - x$, $(0, 0)$

(c) $g(x) = \dfrac{2x + 1}{x - 1}$, $(2, 5)$

(d) $g(x) = \dfrac{1}{\sqrt{x}}$, $(1, 1)$

9. (a) Find the slope of the tangent line to the parabola $y = x^2 + x + 1$ at the general point whose x-coordinate is a.

(b) Find the slopes of the tangents to this parabola at the points whose x-coordinates are -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1.

C **10.** (a) Find the slope of the tangent to the parabola $y = 3x^2 + 2x$ at the point whose x-coordinate is a.

(b) At what point on the parabola is the tangent line parallel to the line $y = 10x - 2$?

11. Find the points of intersection of the parabolas $y = \frac{1}{2}x^2$ and $y = 1 - \frac{1}{2}x^2$. Show that at each of these points the tangent lines to the two parabolas are perpendicular.

1.5 VELOCITY AND OTHER RATES OF CHANGE

If a car is driven on a highway for three hours and the distance covered is 270 km, then it is easy to find the average velocity:

$$\text{average velocity} = \frac{\text{distance travelled}}{\text{time elapsed}} = \frac{270}{3} = 90 \text{ km/h}$$

But if you watch the speedometer of a car while travelling in city traffic, you will see that the indicator does not stay still for very long; that is, the speed of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Before giving a general definition, let us investigate the situation of a falling ball in the following example.

Example 1 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground. How fast is the ball falling after 3 s?

Solution In trying to solve this problem we use the fact, discovered by Galileo almost three centuries ago, that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This neglects air resistance.) If the distance fallen after t seconds is denoted by $s = f(t)$ and measured in metres, then Galileo's law is expressed by the equation

$$s = f(t) = 4.9t^2$$

The difficulty in finding the velocity after 3 s is that we are dealing with a single instant of time ($t = 3$) so there is no time interval involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 3$ to $t = 3.1$:

$$\begin{aligned}\text{average velocity} &= \frac{\text{distance travelled}}{\text{time elapsed}} \\ &= \frac{\Delta s}{\Delta t} \\ &= \frac{f(3.1) - f(3)}{0.1} \\ &= \frac{4.9(3.1)^2 - 4.9(3)^2}{0.1} \\ &= 29.89 \text{ m/s}\end{aligned}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

time interval	average velocity (m/s)
$3 \leqslant t \leqslant 4$	34.3
$3 \leqslant t \leqslant 3.1$	29.89
$3 \leqslant t \leqslant 3.05$	29.645
$3 \leqslant t \leqslant 3.01$	29.449
$3 \leqslant t \leqslant 3.001$	29.4049

It appears that, as we shorten the time period, the average velocity becomes closer to 29.4 m/s. Let us compute the average velocity over the general time interval $3 \leqslant t \leqslant 3 + h$:

$$\begin{aligned}
\text{average velocity} &= \frac{\Delta s}{\Delta t} \\
&= \frac{f(3+h) - f(3)}{h} \\
&= \frac{4.9(3+h)^2 - 4.9(3)^2}{h} \\
&= \frac{4.9(9 + 6h + h^2 - 9)}{h} \\
&= \frac{4.9(6h + h^2)}{h} \\
&= 29.4 + 4.9h \quad \text{if } h \neq 0
\end{aligned}$$

If the time interval is very short, then h is small, so $4.9h$ is close to 0 and the average velocity is close to 29.4 m/s. The **instantaneous velocity** when $t = 3$ is defined to be the limiting value of these average velocities as h approaches 0. Thus, the (instantaneous) velocity after 3 s is

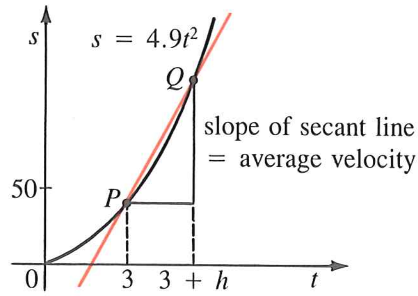
$$\begin{aligned}
v &= \lim_{h \to 0} (29.4 + 4.9h) \\
&= 29.4 \text{ m/s}
\end{aligned}$$

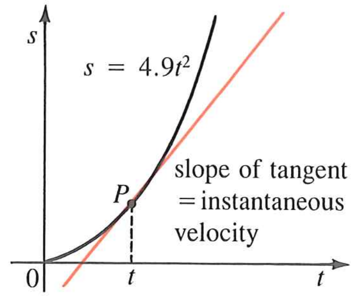
Notice that we did not put $h = 0$ in the expression for the average velocity because that would have resulted in the expression $\frac{0}{0}$, which has no meaning. What we have done is to compute the instantaneous velocity as the *limit* of the average velocities as h approaches 0.

You may have the feeling that the calculations in Example 1 are very similar to those used in finding tangent lines. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball and we consider the points $P(3, 4.9(3)^2)$ and $Q(3+h, 4.9(3+h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(3+h)^2 - 4.9(3)^2}{h}$$

which is the same as the average velocity over the time interval $3 \leqslant t \leqslant 3 + h$ that we found in Example 1. Therefore the velocity at time t (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).





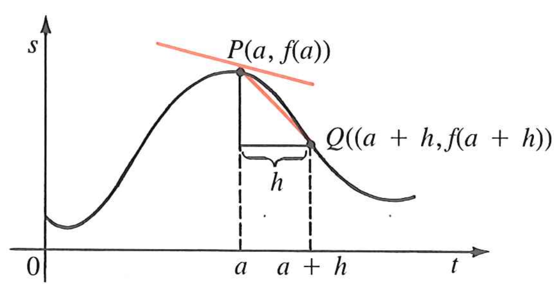
In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$, the change in position is

$$\Delta s = f(a + h) - f(a)$$

The average velocity over this time interval is

$$\frac{\Delta s}{\Delta t} = \frac{f(a + h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ.



Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a+h]$. In other words, we let h approach 0. As in Example 2, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

Instantaneous Velocity

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent at P.

Example 2 The displacement, in metres, of a particle moving in a straight line is given by $s = t^2 + 2t$, where t is measured in seconds. Find the velocity of the particle after 3 s.

Solution If we let $f(t) = t^2 + 2t$, then

$$\begin{aligned} v(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{(3+h)^2 + 2(3+h) - [3^2 + 2(3)]}{h} \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 + 6 + 2h - 15}{h} \\ &= \lim_{h \to 0} \frac{8h + h^2}{h} \\ &= \lim_{h \to 0} (8 + h) = 8 \end{aligned}$$

The velocity after 3 s is 8 m/s.

Other Rates of Change

Suppose that y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $x_1 \leqslant x \leqslant x_2$. By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the **(instantaneous) rate of change of y with respect**

to x at $x = x_1$ and, as with velocity, can be interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$.

$$\text{Rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 3 A thermometer is taken from a room where the temperature is 20°C to the outdoors where the temperature is 5°C. Temperature readings (T) are taken every half-minute and are shown in the following table. The time (t) is measured in minutes.

t	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
T	20	15	12	9.8	8.3	7.2	6.5	6.0	5.7	5.5	5.3

(a) Find the average rate of change of temperature with respect to time over the following time intervals:
(i) $2 \leqslant t \leqslant 4$ (ii) $2 \leqslant t \leqslant 3.5$
(iii) $2 \leqslant t \leqslant 3.0$ (iv) $2 \leqslant t \leqslant 2.5$

(b) Sketch the graph of T as a function of t and use it to estimate the instantaneous rate of change of temperature with respect to time when $t = 2$.

Solution (a) (i) Over the interval $2 \leqslant t \leqslant 4$ the temperature changes from $T = 8.3°$ to $T = 5.7°$, so

$$\Delta T = T(4) - T(2) = 5.7 - 8.3 = -2.6°$$

while the change in time is $\Delta t = 4 - 2 = 2$ min. Therefore the average rate of change of temperature with respect to time is

$$\frac{\Delta T}{\Delta t} = \frac{-2.6}{2} = -1.3°/\text{min}$$

The negative rate of change indicates that the temperature is decreasing.

(ii) $$\frac{\Delta T}{\Delta t} = \frac{T(3.5) - T(2)}{3.5 - 2} = \frac{6 - 8.3}{1.5} = \frac{-2.3}{1.5} \doteq -1.5°/\text{min}$$

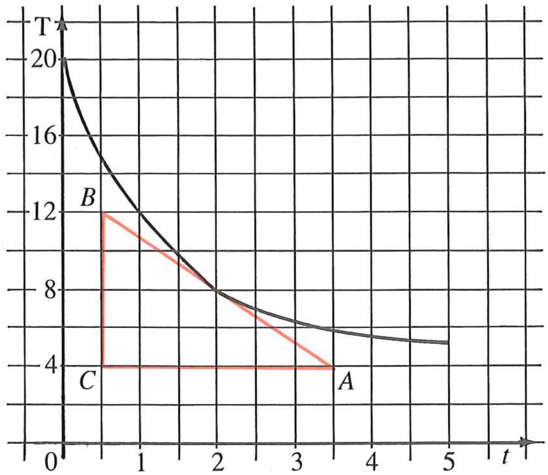
(iii) $$\frac{\Delta T}{\Delta t} = \frac{T(3.0) - T(2)}{3 - 2} = \frac{6.5 - 8.3}{1} = -1.8°/\text{min}$$

(iv) $$\frac{\Delta T}{\Delta t} = \frac{T(2.5) - T(2)}{2.5 - 2} = \frac{7.2 - 8.3}{0.5} = -2.2°/\text{min}$$

(b) We plot the given data and use them to sketch a smooth curve that approximates the graph of the temperature function. Then we draw the tangent at the point P where $x = 2$ and after measuring the sides of triangle ABC, we estimate that the slope of the tangent line is

$$-\frac{BC}{AC} = -\frac{7.5}{3} = -2.5$$

So the rate of change of temperature with respect to time after two minutes is about $-2.5°$/min.



Example 4 A spherical balloon is being inflated. Find the rate of change of the volume with respect to the radius when the radius is 10 cm.

Solution If the radius of the balloon, in centimetres, is r, then the volume, in cubic centimetres, is given by

$$V(r) = \tfrac{4}{3}\pi r^3$$

When $r = 10$, the rate of change of V with respect to r is

$$\begin{aligned}
\lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} &= \lim_{r \to 10} \frac{V(r) - V(10)}{r - 10} \\
&= \lim_{r \to 10} \frac{\frac{4}{3}\pi r^3 - \frac{4}{3}\pi(10)^3}{r - 10} \\
&= \lim_{r \to 10} \tfrac{4}{3}\pi \frac{r^3 - 10^3}{r - 10} \\
&= \lim_{r \to 10} \tfrac{4}{3}\pi \frac{(r - 10)(r^2 + 10r + 10^2)}{r - 10} \quad \text{(difference of cubes)} \\
&= \lim_{r \to 10} \tfrac{4}{3}\pi(r^2 + 10r + 10^2) \\
&= \tfrac{4}{3}\pi[10^2 + 10(10) + 10^2] \\
&= 400\pi \\
&\doteq 1260
\end{aligned}$$

The rate of change of V with respect to r is about 1260 cm^3/cm.

Rates of change occur in all of the sciences. Physicists are interested in the rate of change of displacement with respect to time (called the velocity). Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the rate of reaction). A textile manufacturer is interested in the rate of change of the cost of producing x square metres of fabric per day with respect to x (called the marginal cost). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. All these rates of change can be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry; we are also implicitly solving a great variety of problems involving rates of change in the natural and social sciences as well as in engineering.

EXERCISE 1.5

B **1.** If a ball is thrown into the air with a velocity of 30 m/s, its height in metres after t seconds is given by $y = 30t - 4.9t^2$.

(a) Find the average velocity for the time period beginning when $t = 2$ and lasting

(i) 1 s (ii) 0.5 s (iii) 0.1 s (iv) 0.05 s (v) 0.01 s

(b) Find the instantaneous velocity when $t = 2$.

2. The displacement in metres of a particle moving in a straight line is given by $s = t^2 - 4t + 3$, where t is measured in seconds.

(a) Find the average velocity over the following time periods:

(i) $3 \leqslant t \leqslant 5$ (ii) $3 \leqslant t \leqslant 4$

(iii) $3 \leqslant t \leqslant 3.5$ (iv) $3 \leqslant t \leqslant 3.1$

(b) Find the instantaneous velocity when $t = 3$.

(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a).

(d) Draw the tangent line whose slope is the instantaneous velocity in part (b).

3. A particle moves in a straight line with position function $s = 2t^2 + 4t - 5$, where t is measured in seconds and s in metres. Find the velocity of the particle at time $t = a$. Use this expression to find the velocities after 1 s, 2 s, and 3 s.

4. (a) Use the data of Example 3 to find the average rate of change of temperature with respect to time over the following time intervals:

(i) $3 \leqslant t \leqslant 5$ (ii) $3 \leqslant t \leqslant 4$

(iii) $1 \leqslant t \leqslant 3$ (iv) $2 \leqslant t \leqslant 3$

(b) Use the graph of T to estimate the instantaneous rate of change of T with respect to t when $t = 3$.

5. The population P of a city from 1982 to 1988 is given in the following table:

Year	**1982**	**1983**	**1984**	**1985**	**1986**	**1987**	**1988**
P (in thousands)	211	219	229	241	255	270	286

(a) Find the average rate of growth

(i) from 1984 to 1988 (ii) from 1984 to 1987

(iii) from 1984 to 1986 (iv) from 1984 to 1985

(b) Estimate the instantaneous rate of growth in 1984 by measuring the slope of a tangent.

6. (a) If $y = \frac{2}{x}$, find the average rate of change of y with respect to x over the interval $3 \leqslant x \leqslant 4$. Illustrate by drawing the graph of the function and the secant line whose slope is equal to this average rate of change.

(b) If $y = \frac{2}{x}$, find the instantaneous rate of change of y with respect to x at $x = 3$. Draw the tangent line whose slope is equal to this rate of change.

7. (a) A cubic crystal is being grown in a laboratory. Find the average rate of change of the volume of the cube with respect to its edge length x, measured in millimetres, when x changes from

(i) 4 to 5 (ii) 4 to 4.1 (iii) 4 to 4.01

(b) Find the instantaneous rate of change when $x = 4$.

8. If a tank holds 1000 L of water, which takes an hour to drain from the bottom of the tank, then the volume V of water remaining in the tank after t minutes is

$$V = 1000\left(1 - \frac{t}{60}\right)^2 \qquad 0 \leqslant t \leqslant 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) after 10 min.

C **9.** If an arrow is shot upward on the moon with a velocity of 50 m/s, its height in metres after t seconds is given by $s = 50t - 0.83t^2$.

(a) Find the average velocity for the time period beginning when $t = 1$ and lasting

(i) 1 s (ii) 0.5 s (iii) 0.1 s (iv) 0.05 s (v) 0.01 s

(b) Find the instantaneous velocity when $t = 1$.

(c) Find the velocity after t seconds.

(d) When will the arrow hit the moon?

(e) With what velocity will the arrow hit the moon?

1.6 INFINITE SEQUENCES

A **sequence** is a list of numbers written in a definite order:

$$t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$$

The number t_1 is called the *first term*, t_2 is the *second term*, and in general t_n is the *nth term*. We will be considering only infinite sequences, namely, those in which each term t_n has a successor t_{n+1}.

For every positive integer n there is a corresponding number t_n, so a sequence can be regarded as a function whose domain is the set of positive integers. But we usually write t_n instead of the function notation $t(n)$ for the value of the function at the number n.

Example 1 List the first five terms of the sequence defined by

$$t_n = \frac{n}{n+1}$$

and draw the graph of the sequence.

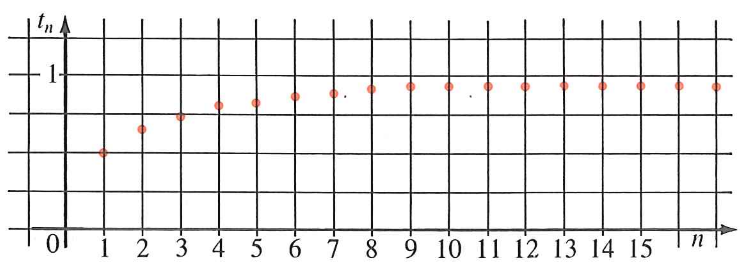
Solution We have

$$t_1 = \tfrac{1}{2}, t_2 = \tfrac{2}{3}, t_3 = \tfrac{3}{4}, t_4 = \tfrac{4}{5}, t_5 = \tfrac{5}{6}$$

This sequence can therefore be described by indicating its initial terms as follows:

$$\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \ldots$$

The graph of this sequence is shown below.



In Example 1, notice that the terms in the sequence are all less than 1 (because $n < n + 1$) but they get closer and closer to 1 as n increases. In fact, we can make the terms t_n as close as we like to 1 by making n large enough. We say the limit of this sequence is 1 and we indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general we say that the sequence with general term t_n has the limit L, and we write

$$\lim_{n\to\infty} t_n = L$$

if the terms t_n are as close as we like to the number L for sufficiently large n.

Example 2 Find $\lim_{n\to\infty} \frac{1}{n}$.

Solution The sequence defined by $t_n = \frac{1}{n}$ is

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

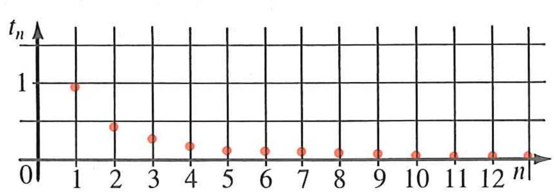
As n becomes larger, $\frac{1}{n}$ becomes smaller. In fact, we can make $\frac{1}{n}$ as close as we like to 0 by making n sufficiently large. For instance, we have

$$\frac{1}{n} < 0.001 \quad \text{if} \quad n > \frac{1}{0.001} = 1000$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

The value of the limit can also be seen from the graph of the sequence.



In the following example and in the exercises we make use of the result of Example 2 and the following more general fact:

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

In addition, we use the fact that the properties of limits stated in Section 1.2 are also valid for limits of sequences.

Example 3 Find $\displaystyle\lim_{n\to\infty} \frac{n^2 - n}{2n^2 + 1}$.

Solution We divide the numerator and denominator by the highest power of n, namely n^2:

$$\frac{n^2 - n}{2n^2 + 1} = \frac{\dfrac{n^2 - n}{n^2}}{\dfrac{2n^2 + 1}{n^2}} = \frac{1 - \dfrac{1}{n}}{2 + \dfrac{1}{n^2}}$$

Thus

$$\lim_{n\to\infty} \frac{n^2 - n}{2n^2 + 1} = \lim_{n\to\infty} \frac{1 - \dfrac{1}{n}}{2 + \dfrac{1}{n^2}}$$

$$= \frac{\lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \dfrac{1}{n}}{\lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty} \dfrac{1}{n^2}}$$

$$= \frac{1 - 0}{2 + 0}$$

$$= \frac{1}{2}$$

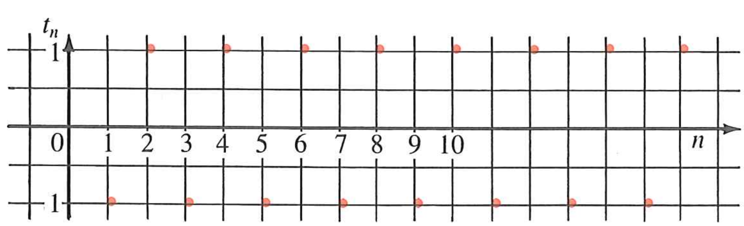
Example 4 Find the following limits, if they exist.

(a) $\displaystyle\lim_{n\to\infty} (-1)^n$ (b) $\displaystyle\lim_{n\to\infty} \left(\tfrac{1}{2}\right)^n$

Solution (a) The terms of the sequence $t_n = (-1)^n$ are

$$-1, 1, -1, 1, -1, 1, \ldots$$

As n increases, the terms do not approach any particular number. They oscillate between -1 and 1 indefinitely, as shown in the graph.



Therefore $\lim_{n \to \infty} (-1)^n$ does not exist.

(b) The terms of the geometric sequence

$$t_n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$$

are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

The denominators 2^n become large as n increases, so

$$\lim_{n \to \infty} \frac{1}{2^n} = 0$$

By the same type of reasoning as in Example 4(b) we have the following result.

If $|r| < 1$, then $\lim_{n \to \infty} r^n = 0$.

Some sequences do not have a simple defining equation but are defined recursively; that is, terms are defined by using preceding terms of the sequence as in the following example.

Example 5 The **Fibonacci sequence** is defined recursively by the equations

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \qquad (n \geq 3)$$

Find the first eight terms of the sequence.

Solution The first two terms are given; the remaining terms are calculated by adding the two preceding terms.

$$\begin{aligned}
f_1 &= 1 \\
f_2 &= 1 \\
f_3 &= f_2 + f_1 = 1 + 1 = 2 \\
f_4 &= f_3 + f_2 = 2 + 1 = 3 \\
f_5 &= f_4 + f_3 = 3 + 2 = 5 \\
f_6 &= 5 + 3 = 8 \\
f_7 &= 8 + 5 = 13 \\
f_8 &= 13 + 8 = 21
\end{aligned}$$

The Fibonacci sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Question 7 in the Exercise).

The idea of a sequence having a limit is implicit in the decimal representation of real numbers. For instance, if we let t_n be the number obtained from the decimal representation of π by truncating after the nth decimal place, then

$$\begin{aligned} t_1 &= 3.1 \\ t_2 &= 3.14 \\ t_3 &= 3.141 \\ t_4 &= 3.141\ 5 \\ t_5 &= 3.141\ 59 \\ t_6 &= 3.141\ 592 \end{aligned}$$

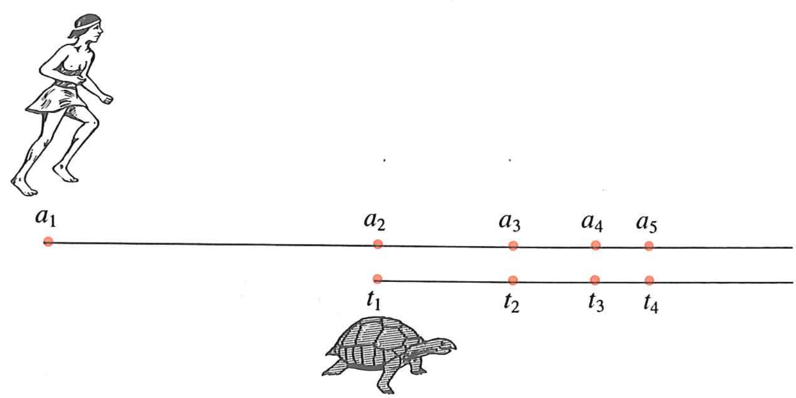
and $$\lim_{n\to\infty} t_n = \pi$$

The terms in this sequence are rational approximations to π.

Zeno's Paradoxes

In the fifth century B.C. the Greek philosopher Zeno of Elea posed four problems, now known as Zeno's paradoxes. These problems were intended to challenge some of the ideas about space and time held at that time.

Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise, who was given a head start. Zeno argued that Achilles could never pass the tortoise. His argument runs this way: Suppose that Achilles starts at position a_1 and the tortoise starts at position t_1. When Achilles reaches the point $a_2 = t_1$, the tortoise is further ahead, at position t_2. When Achilles reaches $a_3 = t_2$, the tortoise is at t_3. This process continues indefinitely, so it appears that the tortoise will always be ahead! But this defies common sense.



One way of explaining the paradox is through the idea of a sequence. The successive positions of Achilles $(a_1, a_2, a_3, \dots)$ and the successive positions of the tortoise ($t_1, t_2, t_3, \ldots$) form sequences $\{a_n\}$ and $\{t_n\}$ where $a_n < t_n$ for all n. It can be shown that both sequences have the same limit:

$$\lim_{n\to\infty} a_n = p = \lim_{n\to\infty} t_n$$

It is precisely at this point p that Achilles overtakes the tortoise.

EXERCISE 1.6

A **1.** State the limits of the following sequences, or state that the limit does not exist.

(a) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots, \left(\frac{1}{3}\right)^n, \ldots$

(b) $5, 4\frac{1}{2}, 4\frac{1}{3}, 4\frac{1}{4}, 4\frac{1}{5}, \ldots, 4 + \frac{1}{n}, \ldots$

(c) $1, 2, 3, 4, 5, \ldots, n, \ldots$

(d) $3, 3, 3, 3, 3, \ldots, 3, \ldots$

(e) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$

(f) $5, 6\frac{1}{2}, 5\frac{2}{3}, 6\frac{1}{4}, 5\frac{4}{5}, 6\frac{1}{6}, \ldots, 6 + \frac{(-1)^n}{n}, \ldots$

(g) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots$

B **2.** List the first six terms of the following sequences.

(a) $t_n = \dfrac{n-1}{2n-1}$ (b) $t_n = \dfrac{2n}{n^2+1}$

(c) $t_n = n2^n$ (d) $t_n = \dfrac{(-1)^{n-1}}{n}$

(e) $t_1 = 1,\ t_n = \dfrac{1}{1 + t_{n-1}}\ (n \geqslant 2)$

(f) $t_1 = 1,\ t_2 = 2,\ t_n = t_{n-1} - t_{n-2}\ (n \geqslant 3)$

3. Find the following limits or state that the limit does not exist.

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^2}$ (b) $\displaystyle\lim_{n\to\infty} \frac{1}{5+n}$

(c) $\displaystyle\lim_{n\to\infty} \left(6 + \frac{1}{n^3}\right)$ (d) $\displaystyle\lim_{n\to\infty} \frac{n}{3n-1}$

(e) $\displaystyle\lim_{n\to\infty} \frac{6n+9}{3n-2}$ (f) $\displaystyle\lim_{n\to\infty} 5n$

(g) $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{2n^2-1}$ (h) $\displaystyle\lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)}$

(i) $\displaystyle\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$ (j) $\displaystyle\lim_{n\to\infty} \left(-\frac{1}{4}\right)^n$

(k) $\lim_{n\to\infty} \frac{n}{n^2+1}$ (l) $\lim_{n\to\infty} (-1)^{n-1} n$

(m) $\lim_{n\to\infty} 5^{-n}$ (n) $\lim_{n\to\infty} (n^3 + n^2)$

(o) $\lim_{n\to\infty} \frac{1+n-2n^2}{1-n+n^2}$ (p) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$

(q) $\lim_{n\to\infty} \frac{1}{n^5}$ (r) $\lim_{n\to\infty} \frac{1-n^3}{1+2n^3}$

(s) $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n$ (t) $\lim_{n\to\infty} \left(\frac{4}{3}\right)^n$

4. If $t_1 = 0.3$, $t_2 = 0.33$, $t_3 = 0.333$, $t_4 = 0.3333$, and so on, what is $\lim_{n\to\infty} t_n$?

5. If

$$t_n = \frac{2^n}{n^2}$$

use your calculator to find t_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Does the limit

$$\lim_{n\to\infty} \frac{2^n}{n^2}$$

exist? If so, guess its value.

6. If

$$t_n = \sqrt[n]{n}$$

use your calculator to find t_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 50, 100, 500, 1000$, and 10 000. Then guess the value of the limit

$$\lim_{n\to\infty} \sqrt[n]{n}$$

7. Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age two months. If we start with one newborn pair, how many pairs of rabbits will there be in the nth month? Show that the answer is f_n, the nth term of the Fibonacci sequence defined in Example 5.

C **8.** Find the limit of the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

by expressing each term as a power of 2.

9. (a) A sequence is defined recursively by

$$t_1 = 1, \; t_n = \frac{1}{2t_{n-1}+1} \qquad (n \geqslant 2)$$

Find t_2, t_3, t_4, t_5, t_6 and guess the value of $\lim_{n\to\infty} t_n$.

(b) Assume that $\lim_{n\to\infty} t_n = L$ exists. What is the value of $\lim_{n\to\infty} t_{n-1}$? Find the value of L by taking the limit of both sides of the recursion equation.

1.7 INFINITE SERIES

Does it make sense to talk about adding infinitely many numbers? You might think this is impossible because it would take an infinite amount of time. But there are situations in which we implicitly use infinite sums. For instance, in decimal notation the symbol $0.\overline{4} = 0.444\,444\,444\ldots$ means

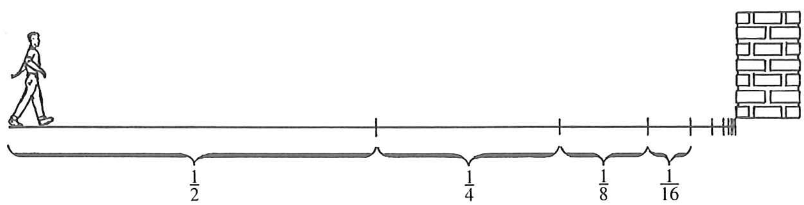
$$\frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10\,000} + \cdots$$

and so, in some sense, it must be true that

$0.\overline{4} = \frac{4}{9}$

$$\frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10\,000} + \cdots = \frac{4}{9}$$

Another situation that gives rise to an infinite sum occurs in one of Zeno's paradoxes, as passed on to us by Aristotle: "A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended."



Of course we know that the man can actually reach the wall, so this suggests that perhaps the total distance can be expressed as the sum of infinitely many smaller distances as follows:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

In order to make sense of this equation, we let S_n be the sum of the first n terms of the series. Then

$$S_1 = \frac{1}{2} = 0.5$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = 0.75$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$$

$$S_5 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} = 0.968\ 75$$

$$S_6 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} = 0.984\ 375$$

$$S_7 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \tfrac{1}{128} = 0.992\ 187\ 5$$

$$\vdots$$

$$S_{10} = \tfrac{1}{2} + \tfrac{1}{4} + \dots + \tfrac{1}{1024} \doteq 0.999\ 023\ 44$$

$$\vdots$$

$$S_{16} = \tfrac{1}{2} + \tfrac{1}{4} + \dots + \tfrac{1}{2^{16}} \doteq 0.999\ 984\ 74$$

Notice that as we add more and more terms, the partial sums become closer and closer to 1. In fact, by making n large enough (that is, by adding sufficiently many terms of the series), we can make the partial sum S_n as close as we like to the number 1. It therefore seems reasonable to say that the sum of the infinite series is 1 and to write

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \dots + \tfrac{1}{2^n} + \dots = 1$$

In other words, the reason the sum of the series is 1 is that

$$\lim_{n\to\infty} S_n = 1$$

We use a similar idea to determine whether or not a general series

$$t_1 + t_2 + t_3 + \dots + t_n + \dots$$

has a sum. We define the **partial sums** as follows.

$$\begin{aligned} S_1 &= t_1 \\ S_2 &= t_1 + t_2 \\ S_3 &= t_1 + t_2 + t_3 \\ S_4 &= t_1 + t_2 + t_3 + t_4 \\ &\vdots \\ S_n &= t_1 + t_2 + t_3 + \dots + t_n \end{aligned}$$

If the infinite sequence $S_1, S_2, \dots, S_n, \dots$ of partial sums of the series $t_1 + t_2 + t_3 + \dots + t_n + \dots$ has a limit L, then we say that the **sum** of the series is L and we write

$$t_1 + t_2 + t_3 + \dots + t_n + \dots = L$$

In sigma notation this becomes

$$\sum_{n=1}^{\infty} t_n = L$$

If a series has a sum, it is called a **convergent** series. If not, it is called **divergent**.

Example 1 Determine whether the following series are convergent or divergent.
(a) $1 + 1 + 1 + 1 + \dots + 1 + \dots$
(b) $1 - 1 + 1 - 1 + \dots + (-1)^{n+1} + \dots$

Solution (a) $S_n = 1 + \underbrace{1 + 1 + \dots + 1}_{n \text{ terms}} = n$

Therefore $\lim_{n \to \infty} S_n = \lim_{n \to \infty} n$ does not exist. It follows that the given series does not have a sum; that is, it is divergent.

(b)
$$S_1 = 1$$
$$S_2 = 1 - 1 = 0$$
$$S_3 = 1 - 1 + 1 = 1$$
$$S_4 = 1 - 1 + 1 - 1 = 0$$

The sequence of partial sums is

$$1, 0, 1, 0, 1, 0, \dots$$

which has no limit. Thus the series $1 - 1 + 1 - 1 + \dots$ does not have a sum; that is, it is divergent.

Example 2 Find the sum of the geometric series

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots \qquad (a \neq 0)$$

when it exists.

Solution The nth partial sum of the geometric series is

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

This is a finite geometric series with first term a and common ratio r. We recall that, for $r \neq 1$, its sum is

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

TAKE CASES

Case 1: If $|r| < 1$, that is, $-1 < r < 1$, we discovered in Section 1.6 that $\lim_{n \to \infty} r^n = 0$. Therefore

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r}$$
$$= \frac{a(1 - 0)}{1 - r}$$
$$= \frac{a}{1 - r}$$

Thus, for $|r| < 1$, the geometric series is convergent and its sum is $\frac{a}{1 - r}$.

Case 2: If $r = 1$, the geometric series becomes

$$a + a + a + a + \ldots$$

which does not have a sum. (See Example 1(a)).

Case 3: If $r = -1$, the geometric series becomes

$$a - a + a - a + \ldots$$

which does not have a sum. (See Example 1(b)).

Case 4: If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist. Therefore $\lim_{n\to\infty} S_n$ does not exist and the geometric series does not have a sum.

We summarize the results of Example 2 as follows:

> If $|r| < 1$, the infinite geometric series
>
> $$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots \qquad (a \neq 0)$$
>
> is convergent and has the sum
>
> $$S = \frac{a}{1 - r}$$
>
> If $|r| \geq 1$, the geometric series is divergent.

In sigma notation we can write

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad |r| < 1$$

Example 3 Find the sum of the series

$$16 - 12 + 9 - \tfrac{27}{4} + \tfrac{81}{16} - \ldots$$

Solution The given series is a geometric series with first term $a = 16$ and common ratio $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series is convergent and its sum is

$$S = \frac{16}{1 - \left(-\frac{3}{4}\right)} = \frac{16}{\frac{7}{4}} = 16 \times \frac{4}{7} = \frac{64}{7}$$

Example 4 Express the repeating decimal $2.1\overline{35}$ as a fraction.

Solution

$$\begin{aligned} 2.1\overline{35} &= 2.135\ 353\ 535\ \ldots \\ &= 2.1 + \frac{35}{1000} + \frac{35}{100\ 000} + \frac{35}{10\ 000\ 000} + \ldots \end{aligned}$$

After the first term, the series is a geometric series with

$$a = \frac{35}{1000} \quad \text{and} \quad r = \frac{1}{100} = 0.01$$

Therefore

$$\begin{aligned} 2.1\overline{35} &= 2.1 + \frac{\frac{35}{1000}}{1 - 0.01} \\ &= 2.1 + \frac{35}{1000(0.99)} \\ &= \frac{21}{10} + \frac{35}{990} \\ &= \frac{2114}{990} \\ &= \frac{1057}{495} \end{aligned}$$

EXERCISE 1.7

B **1.** Find the sum of each of the following series or state that the series is divergent.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ (b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$

(c) $\frac{1}{4} - \frac{5}{16} + \frac{25}{64} - \frac{125}{256} + \ldots$ (d) $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \ldots$

(e) $1 - 2 + 4 - 8 + \ldots$ (f) $60 + 40 + \frac{80}{3} + \frac{160}{9} + \ldots$

(g) $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$

(h) $-3 + 3 - 3 + 3 - 3 + \ldots$

2. Find the sum of each of the following series.

(a) $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$ (b) $\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^{n}$

3. Express the following repeating decimals as fractions.

(a) $0.\overline{1}$ (b) $0.\overline{25}$

(c) $0.4\overline{1}$ (d) $0.\overline{157}$

(e) $1.1\overline{23}$ (f) $2.3\overline{456}$

(g) $0.429\ \overline{113}$ (h) $6.814\ \overline{72}$

4. For what values of x are the following series convergent? In each case find the sum of the series for those values of x.

(a) $1 + x + x^2 + x^3 + \dots$ (b) $1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \dots$

(c) $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots$

(d) $1 + (x - 4) + (x - 4)^2 + (x - 4)^3 + \dots$

(e) $\sum_{n=1}^{\infty} 2^n x^n$

5. The series

$$1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} + \dots + \frac{(-1)^{n-1}}{n^6} + \dots$$

is not a geometric series. Use your calculator to find the first eight partial sums of this series. Does it appear that this series is convergent? If so, estimate its sum to five decimal places.

C **6.** The series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n + 1)} + \dots$$

is not a geometric series.

(a) Use your calculator to find its first 15 partial sums.

(b) Use the identity

$$\frac{1}{k(k + 1)} = \frac{1}{k} - \frac{1}{k + 1}$$

to find an expression for the nth partial sum S_n.

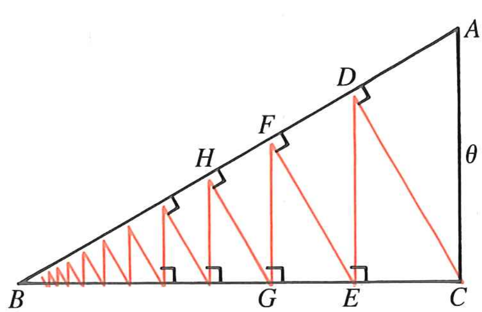
(c) Use part (b) to find the sum of the series.

(d) How many terms of the series would be required so that the partial sum differs from the total sum by less than 0.001?

7. A right triangle is given with $\angle A = \theta$ and $AC = 1$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, EF is perpendicular to AB, and this process is continued indefinitely as in the figure. Find the total length of all the perpendiculars

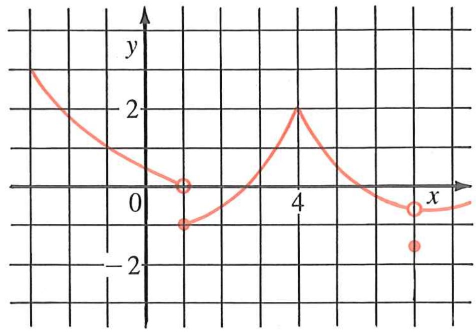
$$CD + DE + EF + FG + \dots$$

in terms of θ.



1.8 REVIEW EXERCISE

1. Use the given graph of f to state the value of the limit, if it exists.



(a) $\lim_{x \to -1} f(x)$ (b) $\lim_{x \to 1^-} f(x)$

(c) $\lim_{x \to 1^+} f(x)$ (d) $\lim_{x \to 1} f(x)$

(e) $\lim_{x \to -3^+} f(x)$ (f) $\lim_{x \to 4^-} f(x)$

(g) $\lim_{x \to 4^+} f(x)$ (h) $\lim_{x \to 4} f(x)$

2. State whether the function f, whose graph is shown in Question 1, is continuous or discontinuous at the following numbers.

(a) 1 (b) 4 (c) 7

3. Find the following limits.

(a) $\lim_{x \to 2} (3x^3 + 7x - 16)$ (b) $\lim_{x \to -1} \dfrac{2x + 3}{3x + 2}$

(c) $\lim_{x \to 2} \dfrac{x^2 - 2x - 8}{x^2 - 7x + 12}$ (d) $\lim_{x \to 4} \dfrac{x^2 - 2x - 8}{x^2 - 7x + 12}$

(e) $\lim_{x \to 5} \sqrt{\dfrac{x^2 - 25}{x - 5}}$ (f) $\lim_{x \to 4} \dfrac{x - 4}{x^3 - 64}$

(g) $\lim_{t \to 0} \dfrac{\sqrt{2 + t} - \sqrt{2}}{t}$ (h) $\lim_{h \to 0} \dfrac{(-3 + h)^2 - 9}{h}$

4. Find the following limits, or state that they do not exist.

(a) $\lim_{x \to -1} \dfrac{x - 6}{(x + 1)^3}$ (b) $\lim_{x \to 1} \dfrac{x^2 - 1}{x^2 - 7x + 6}$

(c) $\lim_{h \to 0} \dfrac{\dfrac{4}{2 + h} - 2}{h}$ (d) $\lim_{y \to 2} \dfrac{y^4 - 16}{y^4 + 2y^3 - y^2 - 2y}$

(e) $\lim_{t \to -2^+} \sqrt[4]{8 + t^3}$ (f) $\lim_{x \to 1^+} \dfrac{|x - 1|}{x - 1}$

(g) $\lim_{x \to 1^-} \dfrac{|x - 1|}{x - 1}$ (h) $\lim_{x \to 1} \dfrac{|x - 1|}{x - 1}$

5. Let
$$f(x) = \begin{cases} -1 - x & \text{if } x < -1 \\ x^2 & \text{if } x \geqslant -1 \end{cases}$$
(a) Find the following limits, if they exist.
(i) $\lim_{x \to -1^-} f(x)$ (ii) $\lim_{x \to -1^+} f(x)$ (iii) $\lim_{x \to -1} f(x)$
(b) Sketch the graph of f.

6. Let
$$g(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leqslant x \leqslant 1 \\ 1 + 2x - x^2 & \text{if } x > 1 \end{cases}$$
(a) Find the following limits, if they exist.
(i) $\lim_{x \to 0^-} g(x)$ (ii) $\lim_{x \to 0^+} g(x)$ (iii) $\lim_{x \to 0} g(x)$
(iv) $\lim_{x \to 1^-} g(x)$ (v) $\lim_{x \to 1^+} g(x)$ (vi) $\lim_{x \to 1} g(x)$
(b) Sketch the graph of g.
(c) Where is g discontinuous?

7. A daytime coin-paid phone call from Toronto to Montreal costs \$1.95 for the first minute and \$0.45 for each additional minute (or part of a minute). Draw the graph of the cost C (in dollars) of the phone call as a function of the time t (in minutes). For what values of t does this function have discontinuities?

8. The point $P(1, -2)$ lies on the curve $y = x^3 - 3x$.
(a) If Q is the point $(x, x^3 - 3x)$, find the slope of the secant line PQ for the following values of x:
(i) 2 (ii) 1.5 (iii) 1.1 (iv) 1.01
(b) Find the slope of the tangent line to the curve at P.
(c) Find an equation of the tangent line to the curve at P.
(d) Graph the curve and the tangent line.

9. Find the equation of the tangent line to the curve $y = x^4$ at the point $(-1, 1)$.

10. If a stone is dropped off a 200 m high cliff, then its height after t seconds, and before it hits the ground, is $h = 200 - 4.9t^2$.
(a) Find the average velocity of the stone for the following time periods.
(i) $1 \leqslant t \leqslant 2$ (ii) $1 \leqslant t \leqslant 1.1$
(b) Find the instantaneous velocity when $t = 1$.

11. A spherical balloon is being inflated. Find the rate of change of the surface area of the balloon with respect to the radius when the radius is 10 cm. (Use the formula $S = 4\pi r^2$, where r is the radius of a sphere and S is the surface area.)

12. Find the following limits or state that the limit does not exist.
(a) $\lim_{n \to \infty} \left(2 - \frac{1}{n} + \frac{3}{n^2}\right)$ (b) $\lim_{n \to \infty} \frac{1 + 2n}{1 - 3n}$
(c) $\lim_{n \to \infty} (1.1)^n$ (d) $\lim_{n \to \infty} \frac{3^n}{5^n}$

13. Find the sum of the series or state that it is divergent.

(a) $6 - 1 + \frac{1}{6} - \frac{1}{36} + \dots$ (b) $\frac{1}{9} + \frac{1}{3} + 1 + 3 + \dots$

14. Express the repeating decimal $1.2\overline{45}$ as a fraction.

15. For what values of x is the series $\sum_{n=1}^{\infty} (x + 1)^n$ convergent? Find the sum of the series for those values of x.

16. A sequence is defined recursively as follows:

$$t_1 = \sqrt{3}, \quad t_{n+1} = \sqrt{3t_n} \quad (n \geq 1)$$

Find $\lim_{n \to \infty} t_n$.

1.9 CHAPTER 1 TEST

1. Find the following limits.

(a) $\lim_{x \to 2} \sqrt{\dfrac{x^2 + 5}{x - 1}}$

(b) $\lim_{x \to -1} \dfrac{x + 1}{x^2 - 4x - 5}$

(c) $\lim_{x \to 1} \dfrac{\dfrac{1}{\sqrt{x}} - 1}{x - 1}$

2. The points $P(2, -1)$ and $Q(3, -4)$ lie on the parabola $y = -x^2 + 2x - 1$.

(a) Find the slope of the secant line PQ.

(b) Find the slope of the tangent line to the parabola at P.

(c) Find the equation of the tangent line at P.

(d) Graph the parabola, the secant line, and the tangent line.

3. Let

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ 2x - 1 & \text{if } x > 0 \end{cases}$$

(a) Find the following limits if they exist.

(i) $\lim_{x \to 0^-} f(x)$ (ii) $\lim_{x \to 0^+} f(x)$ (iii) $\lim_{x \to 0} f(x)$

(b) Sketch the graph of f.

(c) Where is f discontinuous?

4. The displacement in metres of a particle moving in a straight line is given by $s = 5t^2 - 6t + 14$, where t is measured in seconds.

(a) Find the average velocity over the time interval $2 \leqslant t \leqslant 3$.

(b) Find the instantaneous velocity when $t = 2$.

5. Evaluate $\lim_{n \to \infty} \left(\dfrac{1}{8^n} + \dfrac{6n - 2}{2n - 3} \right)$.

6. Find the sum of the series

$$12 - 9 + \frac{27}{4} - \frac{81}{16} + \cdots$$

FOUNDERS OF CALCULUS

Sir Isaac Newton was born in the village of Woolsthorpe, England, on Christmas day in 1642, the year of Galileo's death. The signs of genius did not emerge in high school, but while he was a student at Trinity College, Cambridge, he read the works of Euclid and Descartes and these inspired him. In 1665, Cambridge was closed because of the plague and, while at home on this enforced vacation, Newton made four of his greatest discoveries: the law of gravitation, the nature of light and colour, the method of calculus, and the extended binomial theorem [the expansion of $(a + b)^n$ as an infinite series when n is not a positive integer].

The Greeks had started calculus with their calculations of areas, and mathematicians of the early seventeenth century, such as Fermat and Descartes, had furthered the subject by solving tangent problems. But Newton undertook the first systematic study of calculus. In particular, he was the first to study limits and derivatives, which he called *fluxions*.

His famous book *Principia Mathematica* of 1687 is perhaps the greatest contribution ever made to the mathematical and physical literature. In it, he applied his method of calculus to the theory of gravitation, to hydrostatics and wave motion, and to astronomical problems. He studied the action of the planets on each other, the disturbing action of the sun on the moon, and the variations of the orbit of the moon.

Newton was a professor at Cambridge University and became renowned as the most absent-minded professor of all time. He often forgot to eat meals. This absent-mindedness was a consequence of his extreme powers of concentration. When asked how he was able to solve a difficult problem, he replied: "By always thinking unto it."

Newton was knighted by Queen Anne in 1705. This was the first time that a man of science had been so honoured. When he died in 1727, he was buried in Westminster Abbey with the pomp of a king's funeral. He was very famous in his own time, even to the general public. Alexander Pope wrote

Nature and nature's laws lay hid in night,
God said, "Let Newton be," and all was light.

PROBLEMS PLUS

What is wrong with the following calculation using infinite series?

$$\begin{aligned} 0 &= 0 + 0 + 0 + \dots \\ &= (1 - 1) + (1 - 1) + (1 - 1) + \dots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \dots \\ &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots \\ &= 1 + 0 + 0 + 0 + \dots \\ &= 1 \end{aligned}$$

ANSWERS

CHAPTER 1 LIMITS AND RATES OF CHANGE

REVIEW AND PREVIEW TO CHAPTER 1

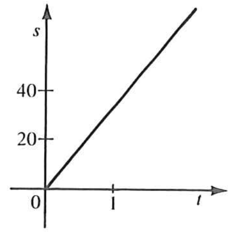
EXERCISE 1

1. (a) $(x-2)(x+1)$ (b) $(x-2)(x-7)$
(c) $(x+3)(x+4)$ (d) $(2x+1)(x-1)$
(e) $(5x+3)(x+2)$ (f) $(3y-1)(2y-3)$
(g) $t(t-1)(t+3)$ (h) $x^2(3x+1)(x+2)$
2. (a) $(2x+5)(2x-5)$
(b) $(x-1)(x^2+x+1)$
(c) $(t+4)(t^2-4t+16)$
(d) $y(y+3)(y-3)$
(e) $(2c-3d)(4c^2+6cd+9d^2)$
(f) $(x^2+2)(x^4-2x^2+4)$
(g) $(x+2)(x-2)(x^2+4)$
(h) $(r+1)(r-1)(r^2+1)(r^4+1)$
3. (a) $(x+4)(x-4)(x-1)$
(b) $(x-1)(x+3)(x-2)$
(c) $(x-2)(x+3)(x+4)$
(d) $(x-3)(x+1)(x+4)$
(e) $(x+2)(2x-1)(2x+3)$
(f) $(x+3)(x-3)(x-2)(x-1)$
4. (a) $x^{\frac{1}{2}}(x-1)(x+1)$ (b) $x^{-1}(x+2)(x+3)$
(c) $x^{-\frac{1}{2}}(x+4)(x-2)$
(d) $2x^{\frac{1}{2}}(x-1)(x^2+x+1)$
(e) $x^{-2}(x+1)^2$ (f) $(x^2+1)^{-\frac{1}{2}}(x^2+4)$

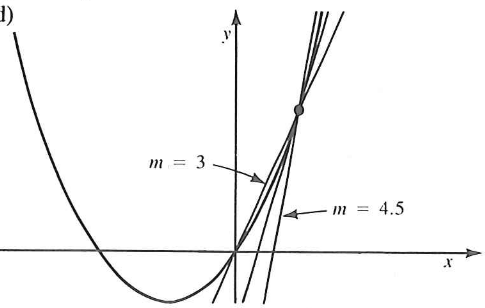
EXERCISE 2

1. (a) $\dfrac{1}{\sqrt{x}+3}$ (b) $-\dfrac{1}{x+\sqrt{x}}$ (c) $\dfrac{x^2+4x+16}{x\sqrt{x}+8}$
(d) $\dfrac{2}{\sqrt{2+h}-\sqrt{2-h}}$
(e) $\dfrac{3x+4}{\sqrt{x^2+3x+4}+x}$
(f) $\dfrac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$
2. (a) $\dfrac{\sqrt{x+1}+1}{x}$ (b) $2(\sqrt{x+2}-\sqrt{x})$
(c) $x(\sqrt{x^2+1}-x)$
(d) $\frac{1}{2}x^2(\sqrt{x+1}+\sqrt{x-1})$

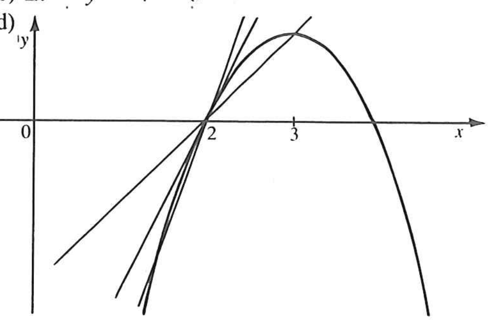
4. (a) increases by 12 (b) decreases by 6
5. (a) decreases by 3 (b) increases by 2
6. $s = 35t$, slope represents speed



7. (a) (i) 5 (ii) 4.5 (iii) 4.1 (iv) 4.01
(v) 4.001 (vi) 3 (vii) 3.5 (viii) 3.9
(ix) 3.99 (x) 3.999 (b) 4
(c) $4x - y - 1 = 0$
(d)



8. (a) (i) 1 (ii) 1.5 (iii) 1.9 (iv) 1.99 (v) 3
(vi) 2.5 (vii) 2.1 (viii) 2.01 (b) 2
(c) $2x - y - 4 = 0$
(d)

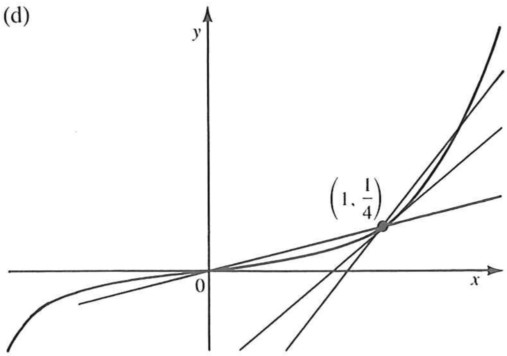


EXERCISE 1.1

1. (a) 4 (b) 3 (c) $\frac{1}{3}$ (d) -3 (e) $-\frac{1}{2}$ (f) $-\frac{1}{2}$
2. $10x + 7y - 5 = 0$ 3. $f(x) = 2x + 6$

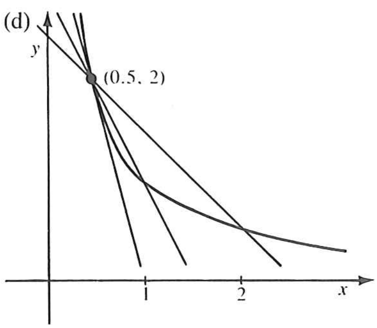
9. (a) (i) 1.75 (ii) 1.1875 (iii) 0.827 5 (iv) 0.757 525 (v) 0.750 75 (vi) 0.25 (vii) 0.4375 (viii) 0.6775 (ix) 0.742 525 (x) 0.749 25 (b) $\frac{3}{4}$ (c) $3x - 4y - 2 = 0$

(d)

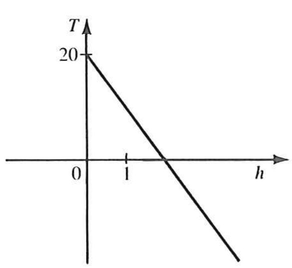


10. (a) (i) -1 (ii) -2 (iii) -2.2222 (iv) -2.5 (v) -2.8571 (vi) -3.3333 (vii) -3.6364 (viii) -3.9216 (ix) -4.4444 (x) -4.0816 (b) -4 (c) $4x + y - 4 = 0$

(d)



11. (a) $T = 20 - 10h$ (h in kilometres)
(b) The slope represents the rate of increase of temperature as the altitude increases.



12. (a) $C = \frac{1}{4}d + 300$ (b) \$800
(c) cost per kilometre for gas, oil, tires, ...
(d) \$300 This is reasonable (insurance, license, depreciation, ...).
(e) Total cost is fixed expenses plus per kilometre expenses.

EXERCISE 1.2

1. (a) 1 (b) 0 (c) 1 (d) does not exist

2. (a) 8 (b) π (c) 3 (d) 2 (d) 2 (e) k^6 (f) π

3. (a) -4 (b) 10 (c) 0 (d) 729 (e) -1 (f) $\frac{7}{6}$ (g) $\frac{11}{4}$ (h) $12\sqrt{2}$ (i) 3 (j) 21

4. (a) $-\frac{1}{4}$ (b) -1 (c) 2 (d) $\frac{1}{2}$ (e) $\frac{3}{2}$ (f) $\frac{1}{27}$ (g) 6 (h) $-\frac{1}{4}$

5. (a) 48 (b) -4 (c) -1 (d) 32 (e) $\frac{1}{6}$ (f) $-\frac{1}{4}$

6. (a) does not exist (b) 0 (c) 4 (d) does not exist (e) does not exist (f) does not exist (g) $-\frac{2}{27}$ (h) $-\frac{1}{16}$ (i) -1 (j) -2

7. (a) 2.000 000, 2.593 742, 2.704 814, 2.716 924, 2.718 146, 2.718 268, 2.718 280, 2.718 282 (b) 2.718 28

8. (a) 1.0000, 0.7177, 0.6956, 0.6934, 0.6932 (b) 0.693

9. (a) 12 (b) $\frac{1}{2}$ **11.** within 0.000 25

13. $f(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x}$

EXERCISE 1.3

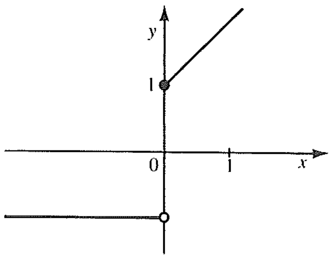
1. (a) 0 (b) 2 (c) 1 (d) does not exist (e) 3 (f) 3 (g) 3 (h) 4

2. (a) 2 (b) 2 (c) 1 (d) does not exist (e) 0 (f) 0 (g) 0 (h) 1

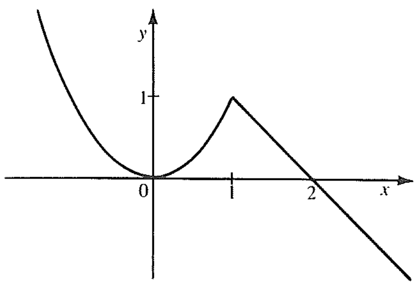
3. (a) continuous (b) discontinuous (c) discontinuous (d) continuous (e) discontinuous

4. (a) 0 (b) 0 (c) 0 (d) 0 (e) 0 (f) 0 (g) 0 (h) 1 (i) -1 (j) does not exist

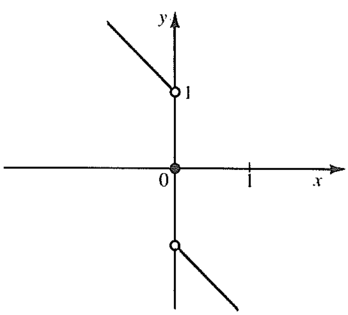
5. (a) -1 (b) 1 (c) does not exist



6. (a) 1 (b) 1 (c) 1

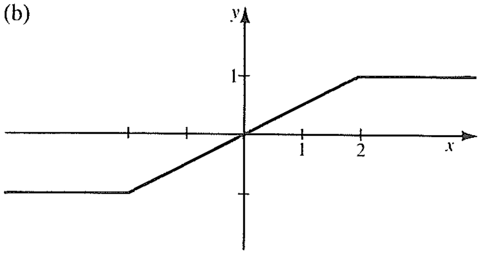


7. (a) 1 (b) -1 (c) does not exist



8. (a) (i) -1 (ii) -1 (iii) 1 (iv) 1

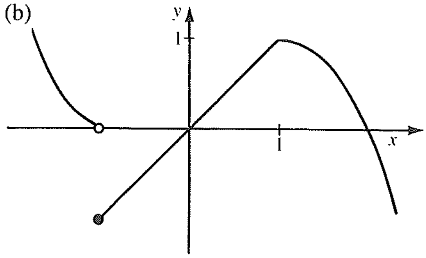
(b)



(c) nowhere

9. (a) (i) 0 (ii) -1 (iii) does not exist (iv) 1
(v) 1 (vi) 1

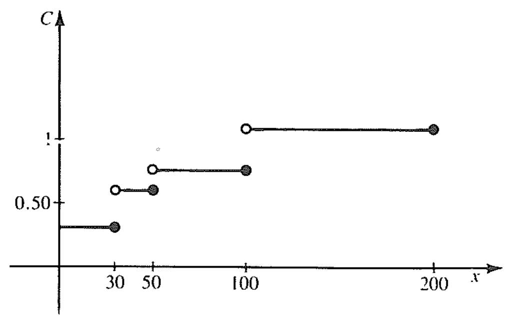
(b)



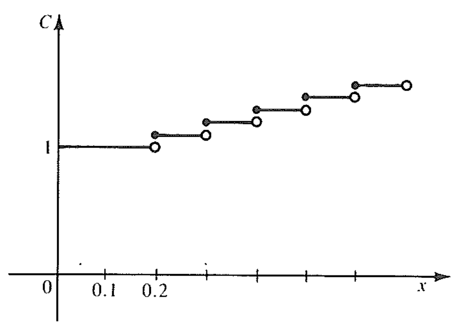
(c) -1

10. (a) 4 (b) 1 (c) -1 (d) 1, 3

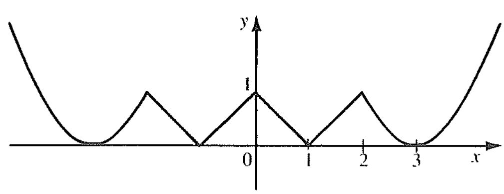
11. Discontinuities at $x = 30, 50, 100$.



12. Discontinuities at $x = 0.2, 0.3, 0.4, 0.5, \ldots$.



13. Continuous everwhere.

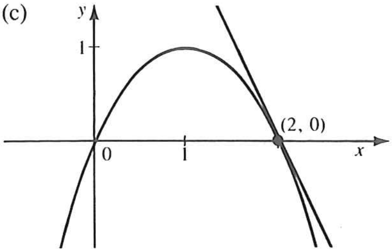


14. $\frac{1}{3}$

EXERCISE 1.4

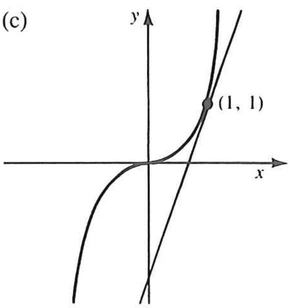
1. (a) -2 (b) $2x + y - 4 = 0$

(c)



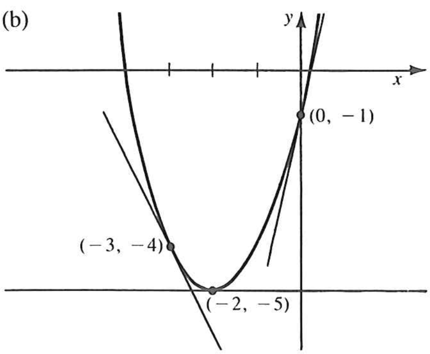
2. (a) 3 (b) $3x - y - 2 = 0$

(c)



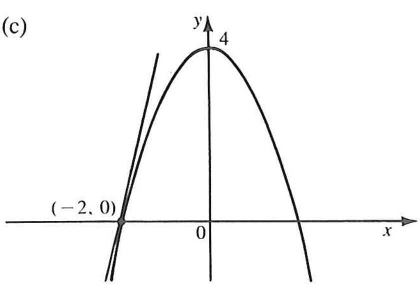
6. (a) (i) -2 (ii) 0 (iii) 4

(b)



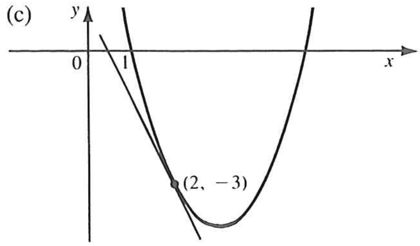
7. (i) (a) 4 (b) $4x - y + 8 = 0$

(c)



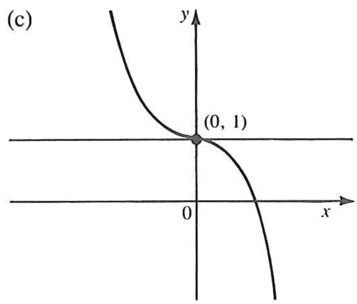
(ii) (a) -2 (b) $2x + y - 1 = 0$

(c)



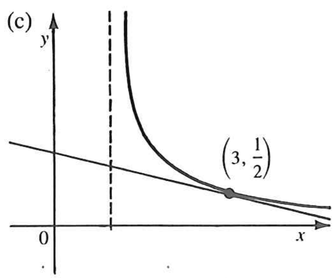
(iii) (a) 0 (b) $y = 1$

(c)



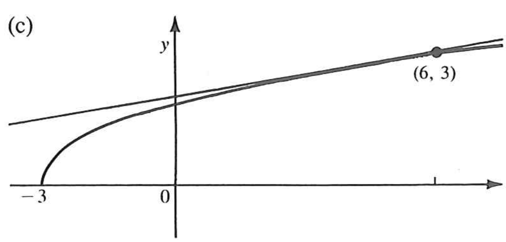
(iv) (a) $-\frac{1}{4}$ (b) $x + 4y - 5 = 0$

(c)



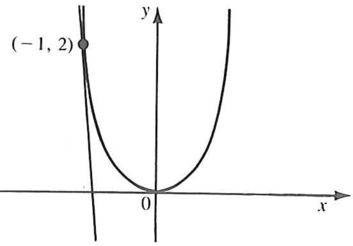
(v) (a) $\frac{1}{6}$ (b) $x - 6y + 12 = 0$

(c)



(vi) (a) -8 (b) $8x + y + 6 = 0$

(c)



8. (a) $7x + y - 1 = 0$ (b) $x + y = 0$
(c) $3x + y - 11 = 0$ (d) $x + 2y - 3 = 0$

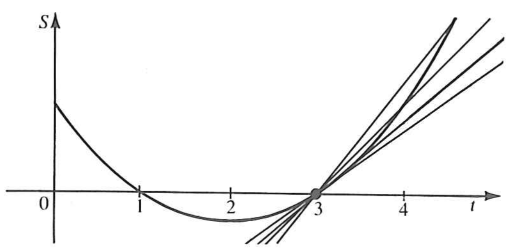
9. (a) $2a + 1$ (b) $-1, 0, 1, 2, 3$

10. (a) $6a + 2$ (b) $\left(\frac{4}{3}, 8\right)$ **11.** $\left(\pm 1, \frac{1}{2}\right)$

EXERCISE 1.5

1. (a) (i) 5.5 m/s (ii) 7.95 m/s (iii) 9.91 m/s
(iv) 10.155 m/s (v) 10.351 m/s (b) 10.4 m/s

2. (a) (i) 4 m/s (ii) 3 m/s (iii) 2.5 m/s
(iv) 2.1 m/s (b) 2 m/s
(c), (d)

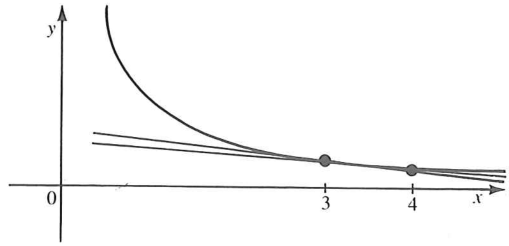


3. $4a + 4$, 8 m/s, 12 m/s, 16 m/s

4. (a) (i) -0.6°/min (ii) -0.8°/min
(iii) -2.75°/min (iv) -1.8°/min
(b) -1°/min

5. (a) (i) 14.3 thousand/year
(ii) 13.7 thousand/year (iii) 13.0 thousand/year
(iv) 12.0 thousand/year (b) 11 thousand/year

6. (a) $-\frac{1}{6}$ (b) $-\frac{2}{9}$



7. (a) (i) 61 mm³/mm (ii) 49.21 mm³/mm
(iii) 48.1201 mm³/mm (b) 48 mm³/mm

8. $\frac{250}{9}$ L/min

9. (a) (i) 47.51 m/s (ii) 47.93 m/s
(iii) 48.26 m/s (iv) 48.30 m/s (v) 48.33 m/s
(b) 48.34 m/s (c) $50 - 1.66t$
(d) After about 60.24 s (e) -50 m/s

EXERCISE 1.6

1. (a) 0 (b) 4 (c) does not exist (d) 3 (e) 0
(f) 6 (g) does not exist

2. (a) $0, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$ (b) $1, \frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}, \frac{12}{37}$
(c) 2, 8, 24, 64, 160, 384
(d) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}$ (e) $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$
(f) $1, 2, 1, -1, -2, -1$

3. (a) 0 (b) 0 (c) 6 (d) $\frac{1}{3}$ (e) 2 (f) does not exist (g) $\frac{1}{2}$ (h) 1 (i) 0 (j) 0 (k) 0
(l) does not exist (m) 0 (n) does not exist
(o) -2 (p) 0 (q) 0 (r) $-\frac{1}{2}$ (s) 0 (t) does not exist **4.** $\frac{1}{3}$

5. $2, 1, \frac{8}{9}, 1, \frac{32}{25}, \frac{16}{9}, \frac{128}{49}, 4, \frac{512}{81}, \frac{1024}{100}$, 2621.4,
4.5×10^{11}, 128×10^{26}; does not exist

6. 1, 1.414 214, 1.442 250, 1.414 214, 1.379 730,
1.348 006, 1.320 469, 1.296 840, 1.276 518,
1.258 925, 1.081 383, 1.047 129, 1.012 507,
1.006 932, 1.000 921; 1

8. 2 **9.** (a) $1, \frac{1}{3}, \frac{3}{5}, \frac{5}{11}, \frac{11}{21}, \frac{21}{43}; \frac{1}{2}$ (b) L, $\frac{1}{2}$

EXERCISE 1.7

1. (a) $\frac{3}{2}$ (b) $\frac{3}{5}$ (c) divergent (d) $\frac{15}{4}$
(e) divergent (f) 180 (g) $\frac{1}{5}$ (h) divergent

2. (a) 8 (b) $-\frac{2}{7}$

3. (a) $\frac{1}{9}$ (b) $\frac{25}{99}$ (c) $\frac{41}{99}$ (d) $\frac{157}{999}$ (e) $\frac{556}{495}$ (f) $\frac{7811}{3330}$
(g) $\frac{107\,171}{249\,750}$ (h) $\frac{37\,481}{5500}$

4. (a) $|x| < 1, \dfrac{1}{1-x}$ (b) $|x| < 3, \dfrac{3}{3-x}$
(c) $|x| > 1, \dfrac{x}{x-1}$ (d) $3 < x < 5, \dfrac{1}{5-x}$
(e) $|x| < \frac{1}{2}, \dfrac{2x}{1-2x}$

5. 1, 0.984 375, 0.985 747, 0.985 503, 0.985 567, 0.985 545, 0.985 554, 0.985 550; yes; 0.985 55

6. (a) 0.5, 0.6667, 0.75, 0.8, 0.8333, 0.8571, 0.875, 0.8889, 0.9, 0.9091, 0.9167, 0.9231, 0.9286, 0.9333, 0.9375; (b) $1 - \dfrac{1}{n+1}$
(c) 1 (d) 1000 **7.** $\dfrac{\sin\theta}{1-\sin\theta}$

1.8 REVIEW EXERCISE

1. (a) 1 (b) 0 (c) -1 (d) does not exist (e) 3 (f) 2 (g) 2 (h) 2

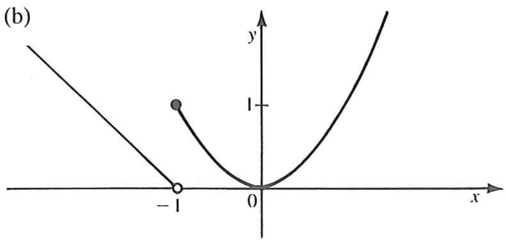
2. (a) discontinuous (b) continuous (c) discontinuous

3. (a) 22 (b) -1 (c) -4 (d) 6 (e) $\sqrt{10}$ (f) $\frac{1}{48}$ (g) $\dfrac{\sqrt{2}}{4}$ (h) -6

4. (a) does not exist (b) $-\frac{2}{5}$ (c) -1 (d) 0 (e) 0 (f) 1 (g) -1 (h) does not exist

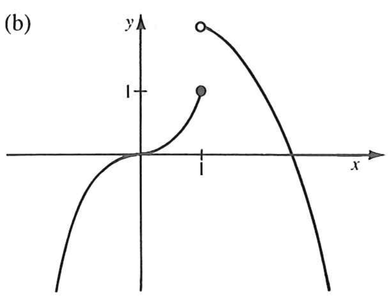
5. (a) (i) 0 (ii) 1 (iii) does not exist

(b)



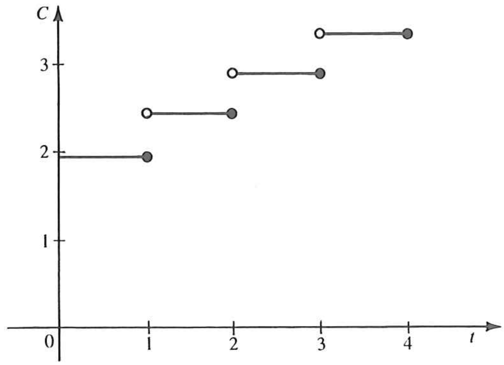
6. (a) (i) 0 (ii) 0 (iii) 0 (iv) 1 (v) 2 (vi) does not exist

(b)



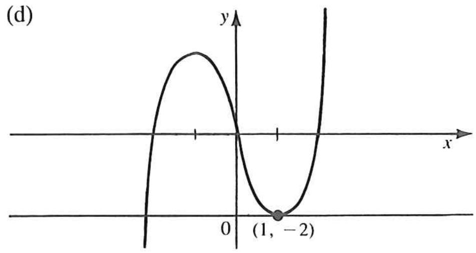
(c) discontinuous at 1

7. $t = 1, 2, 3, 4, 5, \ldots$



8. (a) (i) 4 (ii) 1.75 (iii) 0.31 (iv) 0.0301 (b) 0 (c) $y = -2$

(d)



9. $4x + y + 3 = 0$

10. (a) (i) -14.7 m/s (ii) -10.3 m/s (b) -9.8 m/s

11. 80π cm²/cm **12.** (a) 2 (b) $-\frac{2}{3}$ (c) does not exist (d) 0

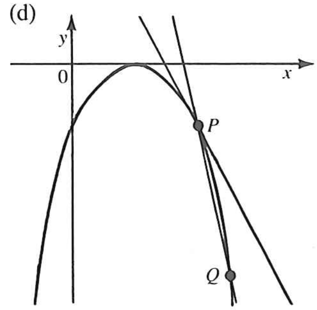
13. (a) $\frac{36}{7}$ (b) divergent

14. $\frac{137}{110}$ **15.** $-2 < x < 0, -1 - \dfrac{1}{x}$ **16.** 3

1.9 CHAPTER 1 TEST

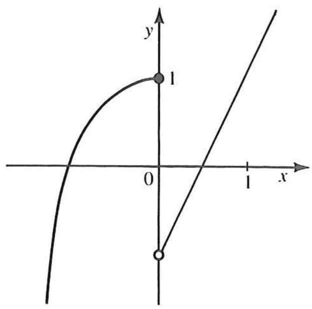
1. (a) 3 (b) $-\frac{1}{6}$ (c) $-\frac{1}{2}$ **2.** (a) -3 (b) -2 (c) $2x + y - 3 = 0$

(d)



3. (a) (i) 1 (ii) -1 (iii) does not exist

(b)



(c) discontinuous at 0

4. (a) 19 m/s (b) 14 m/s **5.** 3 **6.** $\frac{48}{7}$