5. $y = \dfrac{x}{x^2 - 9}$

(a) The domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

(b) The y-intercept is 0 and the x-intercept is 0.

(c) $f(x) = -f(x)$, so the function is odd and it is symmetric about the origin.

(d) $\lim_{x \to \pm\infty} \dfrac{x}{x^2 - 9} = 0$, so there is a horizontal asymptote at $y = 0$.

$\lim_{x \to -3^-} \dfrac{x}{x^2 - 9} = -\infty$, $\lim_{x \to -3^+} \dfrac{x}{x^2 - 9} = \infty$, $\lim_{x \to 3^-} \dfrac{x}{x^2 - 9} = -\infty$, and

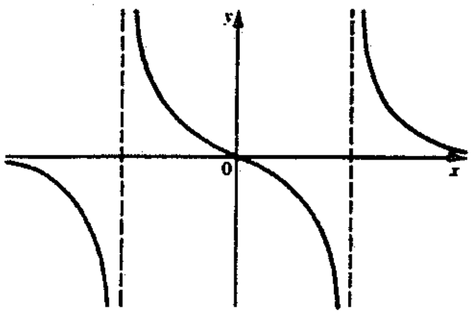
$\lim_{x \to 3^+} \dfrac{x}{x^2 - 9} = \infty$, so there are vertical asymptotes at $x = -3$ and $x = 3$.

(e) $y' = \dfrac{x^2 - 9 - 2x^2}{(x^2 - 9)^2} = -\dfrac{(x^2 + 9)}{(x^2 - 9)} < 0$ on the domain, so y decreases on $(-\infty, -3)$, $(-3, 3)$, and $(3, -\infty)$.

(f) There are no maximum or minimum values.

(g) $y'' = \dfrac{-2x(x^2 - 9) + (x^2 + 9)(4x)}{(x^2 - 9)^3} = \dfrac{2x(x^2 + 27)}{(x^2 - 9)^3}$, so y is CD on $(-\infty, -3)$ and $(0, 3)$ and y is CU on $(-3, 0)$ and $(3, \infty)$. There is a point of inflection at $(0, 0)$.

(h)



EXERCISE 1

1. (a) $\frac{\pi}{6}=\frac{180}{6}=30^\circ$ (b) $\frac{-3\pi}{2}=\frac{-3\times180}{2}=-270^\circ$

(c) $\frac{5\pi}{4}=\frac{5\times180}{4}=225^\circ$ (d) $3\pi=3\times180=540^\circ$

(e) $4=\frac{4\times180}{\pi}\doteq229^\circ$ (f) $-\frac{3}{4}=\frac{-0.75\times180}{\pi}\doteq-43^\circ$

(g) $-12=\frac{-12\times180}{\pi}=-688^\circ$

2. (a) $45^\circ=\frac{45\times\pi}{180}=\frac{\pi}{4}\doteq0.79$ (b) $315^\circ=\frac{315\times\pi}{180}=\frac{7\pi}{4}\doteq5.50$

(c) $-210^\circ=\frac{-210\times\pi}{180}=-\frac{7\pi}{6}\doteq3.67$ (d) $570^\circ=\frac{570\times\pi}{180}=\frac{19\pi}{6}\doteq9.95$

(e) $2^\circ=\frac{2\times\pi}{180}=\frac{\pi}{90}\doteq0.03$ (f) $-28^\circ=\frac{-28\times\pi}{180}=\frac{-7\pi}{45}\doteq-0.49$

(g) $601^\circ=\frac{601\pi}{180}\doteq10.5$

3. (a) $a=r\theta=10\times2.5=25$ (b) $\theta=\frac{a}{r}=\frac{12}{10}=1.2$

4. $72^\circ=\frac{72\times\pi}{180}=\frac{2\pi}{5}=\theta$. Therefore $r=\frac{a}{\theta}=\frac{32}{0.4\pi}=\frac{80}{\pi}\doteq25.46$

EXERCISE 2

1. $r^2=x^2+y^2=9+16=25$. Therefore $r=5$ and
$\sin\theta=\frac{4}{5}$, $\cos\theta=\frac{3}{5}$, $\tan\theta=\frac{4}{3}$, $\csc\theta=\frac{5}{4}$, $\sec\theta=\frac{5}{3}$, $\cot\theta=\frac{3}{4}$.

2. $r^2=4+1=5$. Therefore $r=\sqrt{5}$ and
$\sin\theta=\frac{-1}{\sqrt{5}}$, $\cos\theta=\frac{-2}{\sqrt{5}}$, $\tan\theta=\frac{1}{2}$.

3. $r^2=25+144$. Therefore $r=13$ and
$\csc\theta=-\frac{13}{12}$, $\sec\theta=\frac{13}{5}$, $\cot\theta=-\frac{5}{12}$.

4. Since $y=1$ and $r=3$, $x=\sqrt{9-1}=2\sqrt{2}$. Therefore $\cos\theta=\frac{2\sqrt{2}}{3}$ and $\tan\theta=\frac{1}{2\sqrt{2}}$.

5. Since $x=-1$ and $r=2$, $y=\sqrt{4-1}=\sqrt{3}$.
Therefore $\csc\theta=\frac{2}{\sqrt{3}}$ and $\cot\theta=-\frac{1}{\sqrt{3}}$.

6. Since $y=-5$ and $x=3$, $r=\sqrt{25+9}=\sqrt{34}$.
Therefore $\cos\theta=\frac{3}{\sqrt{34}}$ and $\csc\theta=-\frac{\sqrt{34}}{5}$.

7. Since $x=-5$ and $y=-12$, $r=\sqrt{25+144}=13$.
Therefore $\sec\theta=-\frac{13}{5}$ and $\sin\theta=-\frac{12}{13}$.

8. Since $x=-3$ and $y=4$, $r^2=9+16=25$.
Therefore $\sin^2\theta+\cos^2\theta=\frac{16}{25}+\frac{9}{25}=\frac{25}{25}=1$.

9. Since $y=1$ and $r=2$, $x=-\sqrt{4-1}=-\sqrt{3}$. Therefore $\frac{\sin\theta}{\cos\theta}=\frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}}=-\frac{1}{\sqrt{3}}=\tan\theta$.

10. Since $y=-1$ and $r=2$, $x=\sqrt{4-1}=\sqrt{3}$.
Therefore $1+\tan^2\theta=1+\frac{1}{3}=\left(\frac{2}{\sqrt{3}}\right)^2=\sec^2\theta$.

11. (a) $\frac{1}{\sin\theta}=\frac{1}{\frac{y}{r}}=\frac{r}{y}=\csc\theta$ (b) $\frac{1}{\cos\theta}=\frac{1}{\frac{x}{r}}=\frac{r}{x}=\sec\theta$

(c) $\frac{1}{\tan\theta}=\frac{1}{\frac{y}{x}}=\frac{x}{y}=\cot\theta$ (d) $\frac{\sin\theta}{\cos\theta}=\frac{\frac{y}{r}}{\frac{x}{r}}=\frac{y}{x}=\tan\theta$

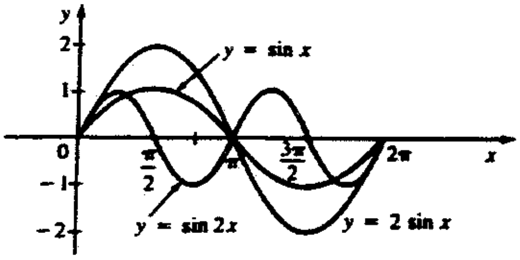
(e) $\frac{\cos\theta}{\sin\theta}=\frac{\frac{x}{r}}{\frac{y}{r}}=\frac{x}{y}=\cot\theta$ (f) $\sin^2\theta+\cos^2\theta=\frac{y^2+x^2}{r^2}=\frac{r^2}{r^2}=1$

12. (a) $1+\tan^2\theta=1+\frac{\sin^2\theta}{\cos^2\theta}=\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}=\sec^2\theta$

(b) $1+\cot^2\theta=1+\frac{\cos^2\theta}{\sin^2\theta}=\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}=\csc^2\theta$

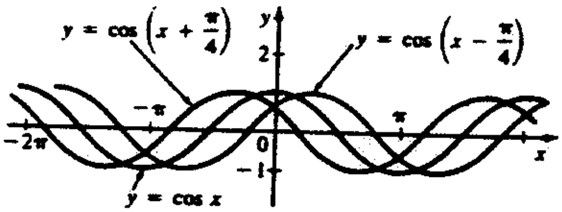
EXERCISE 3

1.

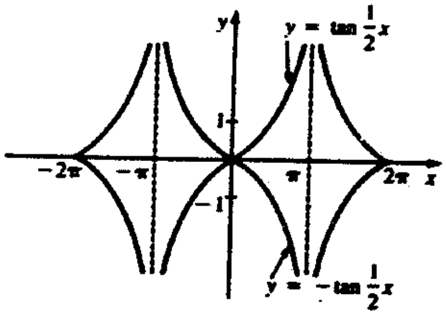


2.

2.



3.



EXERCISE 4

1.(a) $\sin\frac{\pi}{3} - \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = 0$ (b) $\sec\frac{\pi}{6} + 2\cot\frac{\pi}{4} = \frac{2}{\sqrt{3}} + 2(1) = \frac{2+2\sqrt{3}}{\sqrt{3}}$

(c) $\sin^2 30^\circ + \cos^2 45^\circ = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

(d) $4\sin\frac{\pi}{6}+\sec^2\frac{\pi}{4}=4\left(\frac{1}{2}\right)+\left(\sqrt{2}\right)^2=2+2=4$

(e) $\sqrt{3}\cos 30^\circ-\csc 45^\circ+3\sin^2 45^\circ=\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)-\sqrt{2}+3\left(\frac{1}{\sqrt{2}}\right)^2=\frac{3}{2}-\sqrt{2}+\frac{3}{2}=3-\sqrt{2}$

(f) $\tan\frac{\pi}{3}\cos\frac{\pi}{4}\csc\frac{\pi}{3}-\sec\frac{\pi}{6}\tan\frac{\pi}{6}=(\sqrt{3})\left(\frac{1}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{3}}\right)-\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)=\sqrt{2}-\frac{2}{3}$

EXERCISE 6.1

1. (a) $\sin(-\frac{7\pi}{6}) = -\sin(\pi+\frac{\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$

   (b) $\cos\frac{15\pi}{4} = \cos(4\pi-\frac{\pi}{4}) = \cos(-\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

   (c) $\tan(-\frac{8\pi}{3}) = -\tan(\frac{8\pi}{3}) = -\tan(2\pi+\frac{2}{3}\pi) = -\tan(\pi-\frac{\pi}{3}) = \tan\frac{\pi}{3} = \sqrt{3}$

   (d) $\tan\frac{33\pi}{4} = \tan(8\pi+\frac{\pi}{4}) = \tan(\frac{\pi}{4}) = 1$

   (e) $\sin 240^\circ = \sin(180^\circ+60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

   (f) $\cos(-135^\circ) = \cos(180^\circ-45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$

   (g) $\tan 330^\circ = \tan(360^\circ-30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$

   (h) $\sin 495^\circ = \sin(135^\circ) = \sin(180^\circ-45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$

2. (a) $\cos\frac{11\pi}{6} = \cos(\frac{3\pi}{2}+\frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

   (b) $\sin(-\frac{7\pi}{6}) = -\sin\frac{7\pi}{6} = -\sin(\frac{3\pi}{2}-\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}$

   (c) $\sin(120^\circ) = \sin(90^\circ+30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

   (d) $\tan(-\frac{5\pi}{3}) = -\tan\frac{5\pi}{3} = -\tan(\frac{3\pi}{2}+\frac{\pi}{6}) = \cot\frac{\pi}{6} = \sqrt{3}$

   (e) $\tan 510^\circ = \tan 150^\circ = \tan(90^\circ+60^\circ) = -\cot 60^\circ = -\frac{1}{\sqrt{3}}$

   (f) $\cos(-315^\circ) = \cos(270^\circ+45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$

3. (a) $\cos x+\cos(\pi-x)-\cos(\pi+x)-\cos(-x) = \cos x-\cos x+\cos x-\cos x = 0$

   (b) $\tan x+\tan(\pi-x)+\cot(\frac{\pi}{2}-x)-\tan(2\pi-x) = \tan x-\tan x+\tan x+\tan x = 2\tan x$

(c) $\sin(\pi+x)+\cos(\frac{\pi}{2}-x)+\tan(\frac{\pi}{2}+x)+\tan(\frac{3\pi}{2}-x)$
$= -\sin x+\sin x-\cot x+\cot x=0$

(d) $\sin(\frac{\pi}{2}+x)-\cos(\frac{3\pi}{2}-x)+\sin(\frac{3\pi}{2}-x)=\cos x+\sin x-\cos x=\sin x$

(e) $\sin(\frac{\pi}{2}-x)+\sin(\pi-x)+\sin(\frac{3\pi}{2}-x)+\sin(2\pi-x)=\cos x+\sin x-\cos x-\sin x=0$

4. (a) $\csc(\pi-x)=\dfrac{1}{\sin(\pi-x)}=\dfrac{1}{\sin x}=\csc x$

$\sec(\pi-x)=\dfrac{1}{\cos(\pi-x)}=\dfrac{1}{-\cos x}=-\sec x$

$\cot(\pi-x)=\dfrac{1}{\tan(\pi-x)}=\dfrac{1}{-\tan x}=-\cot x$

(b) $\csc(\frac{\pi}{2}+x)=\dfrac{1}{\sin(\frac{\pi}{2}+x)}=\dfrac{1}{\cos x}=\sec x$

$\sec(\frac{\pi}{2}+x)=\dfrac{1}{\cos(\frac{\pi}{2}+x)}=\dfrac{1}{-\sin x}=-\csc x$

$\cot(\frac{\pi}{2}+x)=\dfrac{1}{\tan(\frac{\pi}{2}+x)}=\dfrac{1}{-\cot x}$

(c) $\csc(\pi+x)=\dfrac{1}{\sin(\pi+x)}=\dfrac{1}{-\sin x}=-\csc x$

$\sec(\pi+x)=\dfrac{1}{\cos(\pi+x)}=\dfrac{1}{-\cos x}=-\sec x$

$\cot(\pi+x)=\dfrac{1}{\tan(\pi+x)}=\dfrac{1}{\tan x}=\cot x$

(d) $\csc(\frac{3\pi}{2}+x)=\dfrac{1}{\sin(\frac{3\pi}{2}+x)}=\dfrac{1}{-\cos x}=-\sec x$

$\sec(\frac{3\pi}{2}+x)=\dfrac{1}{\cos(\frac{3\pi}{2}+x)}=\dfrac{1}{\sin x}=\csc x$

$\cot(\frac{3\pi}{2}+x)=\dfrac{1}{\tan(\frac{3\pi}{2}+x)}=\dfrac{1}{-\cot x}$

Exercise 6.1

5. (a) $\sin(x-\pi)=\sin[-(\pi-x)]=-\sin(\pi-x)=-\sin x$

   (b) $\cos(x-\frac{\pi}{2})=\cos[-(\frac{\pi}{2}-x)]=\cos(\frac{\pi}{2}-x)=\sin x$

   (c) $\tan(-x-\pi)=\tan[-(\pi+x)]=-\tan(\pi+x)=-\tan x$

6. (a) $\sec(\pi+\frac{\pi}{3})=-\sec\frac{\pi}{3}=-2$ (b) $\csc(\frac{3\pi}{2}-\frac{\pi}{6})=-\sec\frac{\pi}{6}=-\frac{2}{\sqrt{3}}$

   (c) $\cot(\frac{\pi}{2}+\frac{\pi}{3})=-\tan\frac{\pi}{3}=-\sqrt{3}$ (d) $\sec(\pi-\frac{\pi}{4})=-\sec\frac{\pi}{4}=-\sqrt{2}$

   (e) $\csc(\frac{3\pi}{2}-\frac{\pi}{4})=-\sec\frac{\pi}{4}=-\sqrt{2}$ (f) $\cot(-\pi+\frac{\pi}{4})=-\cot(\pi-\frac{\pi}{4})=\cot\frac{\pi}{4}=1$

7. (a) $\dfrac{\cos(\pi+x)\cos(\frac{\pi}{2}+x)}{\cos(\pi-x)}-\dfrac{\sin(\frac{3\pi}{2}-x)}{\sec(\pi+x)}$

   $=\dfrac{-\cos x(-\sin x)}{-\cos x}-\dfrac{-\cos x}{-\sec x}$

   $=-\sin x-\cos^2 x$

   (b) $\dfrac{\sin(x-\frac{\pi}{2})}{\cos(\pi-x)}+\dfrac{\tan(x-\frac{3\pi}{2})}{-\tan(\pi+x)}$

   $=\dfrac{-\cos x}{-\cos x}+\dfrac{-\cot x}{-\tan x}$

   $=1+\cot^2 x$

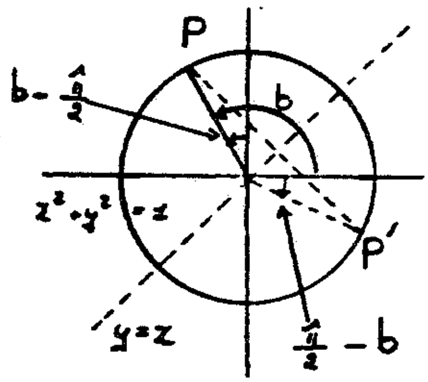
   $=\csc^2 x$

8. $2(1-\sin b\sin c)=(1-\sin b\sin c)+(1-\sin b\sin c)$

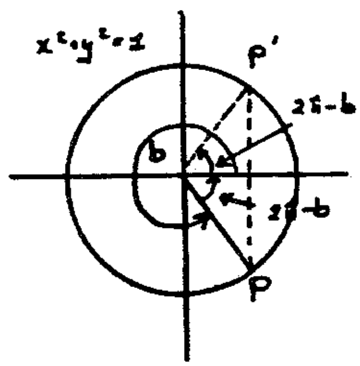
   $=[1-\sin b\sin(\pi-b)][1-\sin(\pi-c)\sin c]$

   $=(1-\sin^2 b)(1-\sin^2 c)=\cos^2 b+\cos^2 c$

9. $\sin B=\sin[\pi-(A+C)]=\sin(A+C)$



10. b determines the point $P(\cos b,\sin b)$ on the unit circle. $P'(\sin b,\cos b)$ is the image of P after a reflection in the line $y=x$. Now P′ is determined by $\frac{\pi}{2}-b$ and has coordinate $P'[\cos(\frac{\pi}{2}-b),\sin(\frac{\pi}{2}-b)]$. Therefore $\sin b=\cos(\frac{\pi}{2}-b)$.



11. b determines the point $P(\cos b,\sin b)$ on the unit circle. $P'(\cos b,-\sin b)$ is the image of P after a reflection in the x-axis. Now P′ is determined by $2\pi-b$ and has coordinate $P'[\cos(2\pi-b),\sin(2\pi-b)]$. Therefore $\sin(2\pi-b)=-\sin b$.

EXERCISE 6.2

1. (a) $\cos 2a \cos a - \sin 2a \sin a = \cos(2a + a) = \cos 3a$

   (b) $\cos x \cos 4x + \sin x \sin 4x = \cos(x - 4x) = \cos(-3x) = \cos 3x$

   (c) $\sin 5 \cos 2 - \cos 5 \sin 2 = \sin(5 - 2) = \sin 3$

   (d) $\sin 2m \cos m + \cos 2m \sin m = \sin(2m + m) = \sin 3m$

   (e) $\dfrac{\tan 2a + \tan 3a}{1 - \tan 2a \tan 3a} = \tan(2a + 3a) = \tan 5a$

   (f) $\dfrac{\tan 7 - \tan 9}{1 + \tan 7 \tan 9} = \tan(7 - 9) = \tan(-2) = -\tan 2$

   (g) $\cos^2 x - \sin^2 x = \cos x \cos x - \sin x \sin x = \cos(x + x) = \cos 2x$

   (h) $\sin a \cos a + \cos a \sin a = \sin(a + a) = \sin 2a$

   (i) $\dfrac{\tan x + \tan x}{1 - \tan^2 x} = \tan(x + x) = \tan 2x$

   (j) $\cos^2 2 + \sin^2 2 = \cos 2 \cos 2 + \sin 2 \sin 2 = \cos(2 - 2) = \cos 0 = 1$

2. (a) $\sin \frac{11\pi}{12} = \sin(\frac{8\pi}{12} + \frac{3\pi}{12}) = \sin \frac{2\pi}{3} \cos \frac{\pi}{4} + \cos \frac{2\pi}{3} \sin \frac{\pi}{4} = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{-1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

   (b) $\cos \frac{13\pi}{12} = \cos(\frac{16\pi}{12} - \frac{3\pi}{12}) = \cos \frac{4\pi}{3} \cos \frac{\pi}{4} + \sin \frac{4\pi}{3} \sin \frac{\pi}{4}$

   $= \frac{-1}{2} \times \frac{1}{\sqrt{2}} + \frac{-\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{-1 - \sqrt{3}}{2\sqrt{2}}$

   (c) $\tan(-\frac{7}{12}\pi) = -\tan(\frac{3\pi}{12} + \frac{4\pi}{12}) = -\dfrac{\tan \frac{\pi}{4} + \tan \frac{\pi}{3}}{1 - \tan \frac{\pi}{4} \tan \frac{\pi}{3}} = -\dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$

   (d) $\tan(-\frac{5}{12}\pi) = \tan(\frac{4\pi}{12} - \frac{9\pi}{12}) = \dfrac{\tan \frac{\pi}{3} - \tan \frac{3\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{3\pi}{4}} = \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}}$

   (e) $\sin 75^\circ = \sin(45 + 30) = \sin 45 \cos 30 + \cos 45 \sin 30 = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$

   (f) $\cos(-15^\circ) = \cos(30 - 45) = \cos 30 \cos 45 + \sin 30 \sin 45 = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$

Exercise 6.2

3. (a) $\sin(\frac{\pi}{4}-\frac{\pi}{3})=\sin\frac{\pi}{4}\cos\frac{\pi}{3}-\cos\frac{\pi}{4}\sin\frac{\pi}{3}=\frac{1}{\sqrt{2}}\times\frac{1}{2}-\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}=\frac{1-\sqrt{3}}{2\sqrt{2}}$

   (b) $\cos(-\frac{\pi}{6}-\frac{\pi}{4})=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$

   (c) $\tan(-\frac{3\pi}{4}+\frac{2\pi}{3})=\frac{\tan\frac{2\pi}{3}-\tan\frac{3\pi}{4}}{1+\tan\frac{2\pi}{3}\tan\frac{3\pi}{4}}=\frac{-\sqrt{3}+1}{1+(-\sqrt{3})(-1)}=\frac{-\sqrt{3}+1}{1+\sqrt{3}}$

4. (a) $\sin(x-y)=\sin x\cos y-\cos x\sin y=\frac{3}{5}\times\frac{12}{13}-\frac{4}{5}\times\frac{5}{13}=\frac{36-20}{65}=\frac{16}{65}$

   (b) $\cos(x+y)=\cos x\cos y-\sin x\sin y=\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}=\frac{48-15}{65}=\frac{33}{65}$

   (c) $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}=\frac{\frac{14}{12}}{\frac{33}{48}}=\frac{56}{33}$

5. (a) $\sin(x+y)=\sin x\cos y+\cos x\sin y=\frac{12}{13}\times\frac{-3}{5}+\frac{-5}{13}\times\frac{-4}{5}=\frac{-36+20}{65}=\frac{-16}{65}$

   (b) $\cos(x-y)=\cos x\cos y+\sin x\sin y=\frac{-5}{13}\times\frac{-3}{5}+\frac{12}{13}\times\frac{-4}{5}=\frac{15-48}{65}=\frac{-33}{65}$

   (c) $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}=\frac{-\frac{12}{5}-\frac{4}{3}}{1-\frac{12}{5}\times\frac{4}{3}}=\frac{-\frac{56}{15}}{-\frac{33}{15}}=\frac{56}{33}$

6. (a) $\sin 50^\circ\cos 20^\circ-\cos 50^\circ\sin 20^\circ=\sin(50-20)=\sin 30=\frac{1}{2}$

   (b) $\cos\frac{\pi}{7}\cos\frac{4\pi}{21}-\sin\frac{\pi}{7}\sin\frac{4\pi}{21}=\cos(\frac{\pi}{7}+\frac{4\pi}{21})=\cos\frac{\pi}{3}=\frac{1}{2}$

   (c) $\frac{\tan 7^\circ+\tan 8^\circ}{1-\tan 7^\circ\tan 8^\circ}=\tan 15=\frac{\tan 45-\tan 30}{1+\tan 45\tan 30}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$

   (d) $\sin\frac{5\pi}{36}\cos\frac{5\pi}{18}+\cos\frac{5\pi}{36}\sin\frac{5\pi}{18}=\sin(\frac{5\pi}{36}+\frac{5\pi}{18})=\sin\frac{5\pi}{12}=\sin(\frac{\pi}{4}+\frac{\pi}{6})$

   $=\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{1+\sqrt{3}}{2\sqrt{2}}$

7. $\sin(a-b)=\sin[a+(-b)]=\sin a\cos(-b)+\cos a\sin(-b)=\sin a\cos b-\cos a\sin b$

8. $\tan(a-b)=\dfrac{\sin(a-b)}{\cos(a-b)}=\dfrac{\sin a\cos b-\cos a\sin b}{\cos a\cos b+\sin a\sin b}$

$$=\frac{\frac{\sin a\cos b}{\cos a\cos b}-\frac{\cos a\sin b}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b}+\frac{\sin a\sin b}{\cos a\cos b}}=\frac{\frac{\sin a}{\cos a}-\frac{\sin b}{\cos b}}{1+\frac{\sin a}{\cos a}\frac{\sin b}{\cos b}}=\frac{\tan a-\tan b}{1+\tan a\tan b}$$

9. $\tan(a-b)=\tan[a+(-b)]=\dfrac{\tan a+\tan(-b)}{1-\tan a\tan(-b)}=\dfrac{\tan a-\tan b}{1+\tan a\tan b}$

10. (a) $\sin(\pi+x)=\sin\pi\cos x+\cos\pi\sin x=0\cos x+(-1)\sin x=-\sin x$

(b) $\tan(2\pi-x)=\dfrac{\tan 2\pi-\tan x}{1+\tan 2\pi\tan x}=\dfrac{0-\tan x}{1+0\tan x}=-\tan x$

(c) $\cos(\frac{3\pi}{2}+x)=\cos\frac{3\pi}{2}\cos x-\sin\frac{3\pi}{2}\sin x=0\cos x-(-1)\sin x=\sin x$

(d) $\sin(\frac{3\pi}{2}-x)=\sin\frac{3\pi}{2}\cos x-\cos\frac{3\pi}{2}\sin x=-\cos x-0\sin x=-\cos x$

(e) $\cos(\frac{\pi}{2}+x)=\cos\frac{\pi}{2}\cos x-\sin\frac{\pi}{2}\sin x=0\cos x-1\sin x=-\sin x$

(f) $\tan(\frac{\pi}{2}+x)=\dfrac{\sin(\frac{\pi}{2}+x)}{\cos(\frac{\pi}{2}+x)}=\dfrac{\sin\frac{\pi}{2}\cos x+\cos\frac{\pi}{2}\sin x}{\cos\frac{\pi}{2}\cos x-\sin\frac{\pi}{2}\sin x}=\dfrac{\cos x}{-\sin x}=-\cot x$

(g) $\sin(x-\pi)=\sin x\cos\pi-\cos x\sin\pi=\sin x(-1)-\cos x(0)=-\sin x$

(h) $-\tan(-x-\pi)=\tan(x+\pi)=\dfrac{\tan x+\tan\pi}{1-\tan x\tan\pi}=\dfrac{\tan x+0}{1-\tan x(0)}=\tan x$

11. In each part let $a+b=x$ and $a-b=y$. Adding we get $2a=x+y\Rightarrow a=\frac{x+y}{2}$. Subtracting we get $2b=x-y\Rightarrow b=\frac{x-y}{2}$.

(a) $\sin x-\sin y$

$=\sin(a+b)-\sin(a-b)=\sin a\cos b+\cos a\sin b-\sin a\cos b+\cos a\sin b$

$=2\cos a\sin b$

$=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

(b) $\cos x+\cos y$

$=\cos(a+b)+\cos(a-b)=\cos a\cos b-\sin a\sin b+\cos a\cos b+\sin a\sin b$

$=2\cos a\cos b$

$=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

(c) $\cos x-\cos y$

$=\cos(a+b)-\cos(a-b)=\cos a\cos b-\sin a\sin b-\cos a\cos b-\sin a\sin b$

$=-2\sin a\sin b$

$=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

12. (a) $\sin 60^\circ+\sin 20^\circ=2\sin 40^\circ\cos 20^\circ$

(b) $\cos 70^\circ-\cos 110^\circ=-2\sin 90\sin(-20)=-2(1)(-\sin 20)=2\sin 20^\circ$

(c) $\cos 40^\circ+\cos 80^\circ=2\cos 60\cos(-20)=2(\frac{1}{2})\cos 20=\cos 20^\circ$

(d) $\sin 6x-\sin 2x=2\cos 4x\sin 2x$

(e) $\sin 130^\circ-\sin 40^\circ=2\cos 85\sin 45=\frac{2}{\sqrt{2}}\cos 85^\circ$

(f) $\cos 4x-\cos 2x=-2\sin 3x\sin x$

13. (a) $\frac{\sin(x-30^\circ)+\cos(60^\circ-x)}{\sin x}=\frac{\sin x\cos 30-\cos x\sin 30+\cos 60\cos x+\sin 60\sin x}{\sin x}$

$=\frac{\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x+\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x}{\sin x}=\frac{\sqrt{3}\sin x}{\sin x}=\sqrt{3}$

(b) $\frac{\tan(\frac{\pi}{4}-x)-\tan(\frac{\pi}{4}+x)}{\tan x}=\frac{\frac{1-\tan x}{1+(1)\tan x}-\frac{1+\tan x}{1-(1)\tan x}}{\tan x}$

$=\frac{(1-\tan x)^2-(1+\tan x)^2}{\tan x(1+\tan x)(1+\tan x)}=\frac{1-2\tan x+\tan^2 x-1-2\tan x-\tan^2 x}{\tan x(1-\tan^2 x)}$

$=\frac{-4\tan x}{\tan x(1-\tan^2 x)}=\frac{-4}{1-\tan^2 x}$

(c) $\frac{\cos 4x+\cos 3x}{\sin 4x-\sin 3x}=\frac{2\cos\frac{7x}{2}\cos\frac{x}{2}}{2\cos\frac{7x}{2}\sin\frac{x}{2}}=\cot\frac{x}{2}$

Exercise 6.2

14. $\sec(x-y)=\dfrac{1}{\cos x\cos y+\sin x\sin y}=\dfrac{1}{\frac{-2\sqrt{2}}{3}\times\frac{2}{5}+\frac{-1}{3}\times\frac{-\sqrt{21}}{5}}$

$=\dfrac{1}{\frac{-4\sqrt{2}+\sqrt{21}}{16}}=\dfrac{15}{\sqrt{21}-4\sqrt{2}}$

15. $\csc(x+y)=\dfrac{1}{\sin x\cos y+\cos x\sin y}=\dfrac{1}{\frac{1}{\csc x\sec y}+\frac{1}{\sec x\csc y}}$

$=\dfrac{\csc x\csc y\sec x\sec y}{\sec x\csc y+\csc x\sec y}$

16. $\sin(x+y+z)=\sin[(x+y)+z]=\sin(x+y)\cos z+\cos(x+y)\sin z$

$=\sin x\cos y\cos z+\cos x\sin y\cos z+\cos x\cos y\sin z-\sin x\sin y\sin z$

17. $\cos(x+y-z)=\cos[(x+y)-z]=\cos(x+y)\cos z+\sin(x+y)\sin z$

$=\cos x\cos y\cos z-\sin x\sin y\cos z+\sin x\cos y\sin z+\cos x\sin y\sin z$

$=\frac{-1}{3}\times\frac{-\sqrt{15}}{4}\times\frac{-2\sqrt{6}}{5}-\frac{2\sqrt{2}}{3}\times\frac{1}{4}\times\frac{-2\sqrt{6}}{5}+\frac{2\sqrt{2}}{3}\times\frac{-\sqrt{15}}{4}\times\frac{1}{5}+\frac{-1}{60}$

$=\dfrac{-2\sqrt{90}+4\sqrt{12}-2\sqrt{30}-1}{60}=\dfrac{-6\sqrt{10}+8\sqrt{3}-2\sqrt{30}-1}{60}$

18. (a) $\sin(x+y)=\sin x\cos y+\cos x\sin y$

$=\sin x\left(\frac{\cos x}{3}\right)+\cos x(2\sin x)=(\frac{1}{3}+2)\sin x\cos x=\frac{7}{3}\sin x\cos x$

(b) $\cos(x+y)=\cos x\cos y-\sin x\sin y$

$=\cos x\left(\frac{\cos x}{3}\right)-\sin x(2\sin x)=\frac{1}{3}\cos^2 x-2(1-\cos^2 x)=\frac{1}{3}\cos^2 x-2+2\cos^2 x$

$=\dfrac{\cos^2 x-6+6\cos^2 x}{3}=\dfrac{7\cos^2 x-6}{3}$

19. $2\sin(x-y)=\sin(x+y)\Rightarrow 2\sin x\cos y-2\cos x\sin y=\sin x\cos y+\cos x\sin y$

$\Rightarrow\ \sin x\cos y=3\cos x\sin y\ \Rightarrow\ \dfrac{\sin x\cos y}{\cos x\cos y}=\dfrac{3\cos x\sin y}{\cos x\cos y}\ \Rightarrow\ \dfrac{\sin x}{\cos x}=3\dfrac{\sin y}{\cos y}$

$\Rightarrow\tan x=3\tan y$ if and only if $\cos y\neq 0$ and $\cos x\neq 0$

Exercise 6.2

20. $\tan(\frac{\pi}{4}+x)=3\tan(\frac{\pi}{4}-x) \Rightarrow \frac{1+\tan x}{1-\tan x}=3\frac{1-\tan x}{1+\tan x} \Rightarrow (1+\tan x)^2=3(1-\tan x)^2$

$\Rightarrow 1+2\tan x+\tan^2 x=3-6\tan x+3\tan^2 x \Rightarrow \tan^2 x-4\tan x+1=0$

Therefore $\tan x=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$

21. (a) $\cos(\frac{3\pi}{4}+x)+\sin(\frac{3\pi}{4}-x)=-\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x=0$

(b) $\cos(\frac{\pi}{12}-x)\sec\frac{\pi}{12}-\sin(\frac{\pi}{12}-x)\csc\frac{\pi}{12}$

$=\left(\cos\frac{\pi}{12}\cos x+\sin\frac{\pi}{12}\sin x\right)\sec\frac{\pi}{12}-\left(\sin\frac{\pi}{12}\cos x-\cos\frac{\pi}{12}\sin x\right)\csc\frac{\pi}{12}$

$=\cos x+\tan\frac{\pi}{12}\sin x-\cos x+\cot\frac{\pi}{12}\sin x$

$=\sin x\left(\tan\frac{\pi}{12}+\cot\frac{\pi}{12}\right)$ where $\tan(\frac{\pi}{12})=\tan(\frac{\pi}{4}-\frac{\pi}{6})=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$

$=\sin x\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)=\sin x\left(\frac{3-2\sqrt{3}+1+3+2\sqrt{3}+1}{3-1}\right)=4\sin x$

(c) $\frac{\sin(x-y)}{\cos x\cos y}+\frac{\sin(z-x)}{\cos z\cos x}+\frac{\sin(y-z)}{\cos y\cos z}$

$=\frac{\sin x\cos y-\cos x\sin y}{\cos x\cos y}+\frac{\sin z\cos x-\cos z\sin x}{\cos z\cos x}+\frac{\sin y\cos z-\cos y\sin z}{\cos y\cos z}$

$=\tan x-\tan y+\tan z-\tan x+\tan y-\tan z=0$

EXERCISE 6.3

1. (a) $\cos 2(2x) = \cos^2 2x - \sin^2 2x = 1 - 2\sin^2 2x = 2\cos^2 2x - 1$

   (b) $\sin 3x = 2\sin \frac{3}{2}x \cos \frac{3}{2}x$

   (c) $\tan 6x = \dfrac{2\tan 3x}{1 - \tan^2 3x}$

   (d) $\sin \frac{1}{2}x = 2\sin \frac{1}{4}x \cos \frac{1}{4}x$

   (e) $\cos \frac{2}{3}x = \cos^2 \frac{1}{3}x - \sin^2 \frac{1}{3}x$

   (f) $\tan(-7x) = \dfrac{2\tan(-\frac{7}{2}x)}{1 - \tan^2(-\frac{7}{2}x)} = \dfrac{-2\tan(\frac{7}{2}x)}{1 - \tan^2(\frac{7}{2}x)}$

2. (a) $2\sin 3\theta \cos 3\theta = \sin 6\theta$

   (b) $6\sin\theta\cos\theta = 3\sin 2\theta$

   (c) $\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{4}\sin\theta$

   (d) $\cos^2\frac{3\theta}{2} - \sin^2\frac{3\theta}{2} = \cos 3\theta$

   (e) $1 - 2\sin^2\frac{\theta}{4} = \cos\frac{\theta}{2}$

   (f) $2\cos^2\frac{7}{2}\theta - 1 = \cos 7\theta$

   (g) $8\sin^2 2\theta - 4 = -4(1 - 2\sin^2 2\theta) = -4\cos 4\theta$

   (h) $1 - 2\sin^2(\frac{\pi}{4} - \frac{x}{2}) = \cos 2(\frac{\pi}{4} - \frac{x}{2}) = \cos(\frac{\pi}{2} - x) = \sin x$

3. $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{3}{5}\times\frac{-4}{5} = \frac{-24}{25}$; $\cos 2\theta = 2\cos^2\theta - 1 = 2\times\frac{16}{25} - 1 = \frac{7}{25}$

   Since $\sin 2\theta < 0$ and $\cos 2\theta > 0$, 2θ is a fourth quadrant angle.

4. $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{12}{13}\times\frac{5}{13} = \frac{120}{169}$. $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2(\frac{144}{169}) = -\frac{119}{169}$

   Since $\sin 2\theta > 0$ and $\cos 2\theta < 0$, 2θ is in the second quadrant.

5. $\sin 4\theta = 2\sin 2\theta\cos 2\theta = 4\sin\theta\cos\theta(1 - 2\sin^2\theta) = 4\times\frac{2}{3}\times\frac{\sqrt{5}}{3}(1 - 2\times\frac{4}{9}) = \frac{8\sqrt{5}}{9}\times\frac{1}{9} = \frac{8\sqrt{5}}{81}$

6. $\csc 2\theta = \dfrac{1}{\sin 2\theta} = \dfrac{1}{2\sin\theta\cos\theta} = \dfrac{1}{2\times\frac{-\sqrt{21}}{5}\times\frac{2}{5}} = \dfrac{25}{-4\sqrt{21}}$

   $\sec 2\theta = \dfrac{1}{\cos 2\theta} = \dfrac{1}{2\cos^2\theta - 1} = \dfrac{1}{\frac{8}{25} - 1} = -\frac{25}{17}$

7. $\tan 2a = \dfrac{2\tan a}{1 - \tan^2 a} = \dfrac{1}{1 - \frac{1}{4}} = \frac{4}{3}$

8. $\tan 4a = \dfrac{2\tan 2a}{1 - \tan^2 2a} = \dfrac{2\left(\frac{2\tan a}{1 - \tan^2 a}\right)}{1 - \left(\frac{2\tan a}{1 - \tan^2 a}\right)^2} = \dfrac{2\times\frac{-4}{3}}{1 - \frac{16}{9}} = \frac{-8}{3}\times\frac{-9}{7} = \frac{72}{21} = \frac{24}{7}$

9. (a) $\sin 3\theta = \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos\theta\cos\theta + (1-2\sin^2\theta)\sin\theta = 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta = 3\sin\theta - 4\sin^3\theta$

(b) $\cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta$

(c) $\tan 3\theta = \tan(2\theta+\theta) = \dfrac{\tan 2\theta + \tan\theta}{1-\tan 2\theta\tan\theta} = \dfrac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan\theta}{1-\tan^2\theta}\times\tan\theta}$

$= \dfrac{\dfrac{2\tan\theta + \tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\dfrac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}} = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(d) $\cos 4\theta = \cos 2(2\theta) = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1$

$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 = 8\cos^4\theta - 8\cos^2\theta + 1$

10. (a) $\cos 135^\circ = 1 - 2\sin^2 67\tfrac{1}{2}^\circ \Rightarrow -\dfrac{1}{\sqrt{2}} = 1 - 2\sin^2 67\tfrac{1}{2}^\circ \Rightarrow 2\sin^2 67\tfrac{1}{2}^\circ = 1 + \dfrac{1}{\sqrt{2}}$

$\Rightarrow \sin^2 67\tfrac{1}{2}^\circ = \dfrac{1}{2} + \dfrac{1}{2\sqrt{2}} \Rightarrow \sin^2 67\tfrac{1}{2}^\circ = \dfrac{\sqrt{2}+1}{2\sqrt{2}} \Rightarrow \sin 67\tfrac{1}{2}^\circ = \sqrt{\dfrac{\sqrt{2}+1}{2\sqrt{2}}}$

(b) $\cos 225^\circ = 2\cos^2 112\tfrac{1}{2}^\circ - 1 \Rightarrow -\dfrac{1}{\sqrt{2}} = 2\cos^2 112\tfrac{1}{2}^\circ - 1 \Rightarrow \cos^2 112\tfrac{1}{2}^\circ = \dfrac{1}{2} - \dfrac{1}{2\sqrt{2}}$

$\Rightarrow \cos^2 112\tfrac{1}{2}^\circ = \dfrac{\sqrt{2}-1}{2\sqrt{2}} \Rightarrow \cos 112\tfrac{1}{2}^\circ = -\sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}$

(c) $\tan 45^\circ = \dfrac{2\tan 22.5}{1-\tan^2 22.5} \Rightarrow 1 = \dfrac{2\tan 22.5}{1-\tan^2 22.5} \Rightarrow 1 - \tan^2 22.5 = 2\tan 22.5$

$\Rightarrow 0 = \tan^2 22.5 + 2\tan 22.5 - 1 \Rightarrow \tan 22.5 = \dfrac{-2\pm\sqrt{4+4}}{2}$

$\Rightarrow \tan 22.5 = -1 + \sqrt{2}$

Exercise 6.3

(d) $\cos(-\frac{\pi}{4})=1-2\sin^2(-\frac{\pi}{8}) \Rightarrow \frac{1}{\sqrt{2}}=1-2\sin^2(-\frac{\pi}{8}) \Rightarrow 2\sin^2(-\frac{\pi}{8})=1-\frac{1}{\sqrt{2}}$

$\Rightarrow \sin^2(-\frac{\pi}{8})=\frac{1}{2}-\frac{1}{2\sqrt{2}} \Rightarrow \sin(-\frac{\pi}{8})=-\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$

(e) $\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}-1 \Rightarrow \frac{1}{\sqrt{2}}+1=2\cos^2\frac{\pi}{8} \Rightarrow \cos\frac{\pi}{8}=\sqrt{\frac{1}{2\sqrt{2}}+\frac{1}{2}}=\sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$

$\cos\frac{\pi}{8}=2\cos^2\frac{\pi}{16}-1 \Rightarrow \cos\frac{\pi}{16}=\sqrt{\frac{\cos\frac{\pi}{8}+1}{2}}=\sqrt{\frac{\sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}+1}{2}}$

(f) $\tan 135=\frac{2\tan 67.5}{1-\tan^2 67.5} \Rightarrow -1=\frac{2\tan 67.5}{1-\tan^2 67.5} \Rightarrow \tan^2 67.5-2\tan 67.5-1=0$

$\tan 67.5=\frac{2+\sqrt{4+4}}{2}=1+\sqrt{2}$

$\tan 67.5=\frac{2\tan 33.75}{1-\tan^2 33.75} \Rightarrow 1+\sqrt{2}=\frac{2\tan 33.75}{1-\tan^2 33.75}$

$(1+\sqrt{2})\tan^2 33.75+2\tan 33.75-(1+\sqrt{2})=0$

$\tan 33.75=\frac{-2+\sqrt{4+4(1+\sqrt{2})^2}}{2(1+\sqrt{2})}=\frac{-2+\sqrt{4+4+8\sqrt{2}+8}}{2(1+\sqrt{2})}=\frac{-1+\sqrt{4+2\sqrt{2}}}{1+\sqrt{2}}$

11. (a) $\cos\theta=1-2\sin^2\frac{\theta}{2} \Rightarrow 2\sin^2\frac{\theta}{2}=1-\cos\theta \Rightarrow \sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$

(b) $\cos\theta=2\cos^2\frac{\theta}{2}-1 \Rightarrow 2\cos^2\frac{\theta}{2}=1+\cos\theta \Rightarrow \cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$

(c) $\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

Exercise 6.3

12. (a) $\sin\frac{x}{2}=+\sqrt{\frac{1-\cos x}{2}}=\sqrt{\frac{1+\frac{12}{13}}{2}}=\sqrt{\frac{25}{26}}=\frac{5}{\sqrt{26}}$

(b) $\cos\frac{x}{2}=-\sqrt{\frac{1+\cos x}{2}}=-\sqrt{\frac{1-\frac{12}{13}}{2}}=-\sqrt{\frac{1}{26}}=-\frac{1}{\sqrt{26}}$

(c) $\tan\frac{x}{2}=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{1+\frac{12}{13}}{1-\frac{12}{13}}}=-\sqrt{\frac{25}{1}}=-5$

13. (a) $\sin 2\theta=2\sin\theta\cos\theta=\frac{2\sin\theta\cos^2\theta}{\cos\theta}=\frac{2\tan\theta}{\sec^2\theta}=\frac{2\tan\theta}{1+\tan^2\theta}$

(b) $\cos 2\theta=2\cos^2\theta-1=\frac{2}{\sec^2\theta}-1=\frac{2}{1+\tan^2\theta}-1=\frac{2-(1+\tan^2\theta)}{1+\tan^2\theta}=\frac{1-\tan^2\theta}{1+\tan^2\theta}$

14. $\frac{\sin 2x}{1+\cos 2x}=\frac{2\sin\cos x}{1+2\cos^2 x-1}=\frac{\sin x}{\cos x}=\tan x$

15. $\frac{1-\cos 2x}{\sin 2x}=\frac{1-(1-2\sin^2 x)}{2\sin x\cos x}=\frac{2\sin^2 x}{2\sin x\cos x}=\frac{\sin x}{\cos x}=\tan x$

16. $\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}}=\frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}=\frac{\cos 2x}{1}=\cos 2x$

17. $\cos\theta+\sin\theta=\frac{2}{3}\Rightarrow(\cos\theta+\sin\theta)^2=\frac{4}{9}\Rightarrow\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta=\frac{4}{9}$
$1+\sin 2\theta=\frac{4}{9}\Rightarrow\sin 2\theta=-\frac{5}{9}$

18. If $\cos\theta+\sin\theta=\frac{1+\sqrt{3}}{2}$ and $\cos\theta-\sin\theta=\frac{1-\sqrt{3}}{2}$
Add the two equations; $2\cos\theta=1\Rightarrow\cos\theta=\frac{1}{2}$
Subtract the two equations; $2\sin\theta=\sqrt{3}\Rightarrow\sin\theta=\frac{\sqrt{3}}{2}$

$\sin 2\theta=2\sin\theta\cos\theta=2\times\frac{\sqrt{3}}{2}\times\frac{1}{2}=\frac{\sqrt{3}}{2}$

Exercise 6.3

19. $2a = \frac{\pi}{2} - b \Rightarrow \cos 2a = \cos(\frac{\pi}{2} - b) \Rightarrow 2\cos^2 a - 1 = \sin b \Rightarrow \cos^2 a = \dfrac{\sin b + 1}{2} \Rightarrow$

$\cos a = \pm\sqrt{\frac{1 + \sin b}{2}}$

20. $\tan 2a = \dfrac{2\tan a}{1 - \tan^2 a} = \dfrac{\frac{2}{5}}{\frac{24}{25}} = \frac{5}{12}$ $\qquad$ $\tan 4a = \dfrac{2\tan 2a}{1 - \tan^2 2a} = \dfrac{\frac{5}{6}}{\frac{119}{144}} = \frac{120}{119}$

$\tan(4a - b) = \dfrac{\tan 4a - \tan b}{1 + \tan 4a \tan b} = \dfrac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}} = \frac{120 \times 239 - 119}{119 \times 239 + 120} = \frac{28\,561}{28\,561} = 1$

21. $\sec 4\theta - \sec 2\theta = 2 \Rightarrow \dfrac{1}{\cos 4\theta} - \dfrac{1}{\cos 2\theta} = 2 \Rightarrow \cos 2\theta - \cos 4\theta = 2\cos 2\theta \cos 4\theta \Rightarrow$

$\cos 2\theta - (2\cos^2 2\theta - 1) = 2\cos 2\theta(2\cos^2 2\theta - 1) \Rightarrow$

$\cos 2\theta - 2\cos^2 2\theta + 1 = 4\cos^3 2\theta - 2\cos 2\theta \Rightarrow 4\cos^3 2\theta + 2\cos^2 2\theta - 3\cos 2\theta - 1 = 0 \Rightarrow$

$(\cos 2\theta + 1)(4\cos^2 2\theta - 2\cos 2\theta - 1) = 0 \Rightarrow$

Therefore $\cos 2\theta = -1 \Rightarrow 2\cos^2\theta - 1 = -1 \Rightarrow \cos^2\theta = 0$ which is not possible within the domain.

Or $\cos 2\theta = \dfrac{2 \pm \sqrt{4 + 16}}{8} = \dfrac{2 \pm 2\sqrt{5}}{8} = \dfrac{1 \pm \sqrt{5}}{4} \Rightarrow 2\cos^2\theta - 1 = \dfrac{1 \pm \sqrt{5}}{4} \Rightarrow$

$\cos^2\theta = \dfrac{5 \pm \sqrt{5}}{8}$

22. $\sin^2(\frac{\pi}{8} + \frac{\theta}{2}) - \sin^2(\frac{\pi}{8} - \frac{\theta}{2})$

$= \left(\sin\frac{\pi}{8}\cos\frac{\theta}{2} + \cos\frac{\pi}{8}\sin\frac{\theta}{2}\right)^2 - \left(\sin\frac{\pi}{8}\cos\frac{\theta}{2} - \cos\frac{\pi}{8}\sin\frac{\theta}{2}\right)^2$

$= 2\sin\frac{\pi}{8}\cos\frac{\theta}{2}\cos\frac{\pi}{8}\sin\frac{\theta}{2} - \left(-2\sin\frac{\pi}{8}\cos\frac{\theta}{2}\cos\frac{\pi}{8}\sin\frac{\theta}{2}\right)$

$= 4\sin\frac{\pi}{8}\cos\frac{\theta}{2}\cos\frac{\pi}{8}\sin\frac{\theta}{2} = 2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\frac{\pi}{4}\sin\theta = \frac{1}{\sqrt{2}}\sin\theta$

EXERCISE 6.4

1. $\sec x - \cos x = \frac{1}{\cos x} - \cos x = \frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \frac{\sin x}{\cos x}\sin x = \sin x \tan x$

2. $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = (1 - \sin^2 x - \sin^2 x)(1) = 1 - 2\sin^2 x$

3. $\csc^2 x + \sec^2 x = \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x} = \csc^2 x \sec^2 x$

4. $\cos^2 x \cos^2 y + \sin^2 x \sin^2 y + \sin^2 x \cos^2 y + \sin^2 y \cos^2 x$
$= \cos^2 x(\cos^2 y + \sin^2 y) + \sin^2 x(\sin^2 y + \cos^2 y) = \cos^2 x + \sin^2 x = 1$

5. $\sec^2 x - \sec^2 y = 1 + \tan^2 x - (1 + \tan^2 y) = \tan^2 x - \tan^2 y$

6. $\frac{\tan x + \tan y}{\cot x + \cot y} = \frac{\tan x + \tan y}{\frac{1}{\tan x} + \frac{1}{\tan y}} = \frac{\tan x + \tan y}{\frac{\tan y + \tan x}{\tan x \tan y}} = \tan x \tan y$

7. $(\sec x - \cos x)(\csc x - \sin x) = \sec x \csc x - \sec x \sin x - \cos x \csc x + \cos x \sin x$

$= \frac{1}{\cos x \sin x} - \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} + \cos x \sin x = \frac{1 - \sin^2 x - \cos^2 x + \cos^2 x \sin^2 x}{\cos x \sin x}$

$= \frac{1 - (\sin^2 x + \cos^2 x) + \cos^2 x \sin^2 x}{\cos x \sin x} = \cos x \sin x$

and $\frac{\tan x}{1 + \tan^2 x} = \frac{\sin x}{\cos x \sec^2 x} = \frac{\sin x \cos^2 x}{\cos x} = \sin x \cos x$

8. $\cos^6 x + \sin^6 x = (\cos^2 x + \sin^2 x)(\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x)$
$= (\cos^4 x + 2\cos^2 x \sin^2 x + \sin^4 x) - 3\cos^2 x \sin^2 x$
$= (\cos^2 x + \sin^2 x)^2 - 3\sin^2 x(1 - \sin^2 x) = 1 - 3\sin^2 x + 3\sin^4 x$

9. $\sec^6 x - \tan^6 x = (\sec^2 x - \tan^2 x)(\sec^4 x + \sec^2 x \tan^2 x + \tan^4 x)$
$= (1)[(\sec^4 x - 2\sec^2 x \tan^2 x + \tan^4 x) + 3\sec^2 x \tan^2 x]$
$= (\sec^2 x - \tan^2 x)^2 + 3\sec^2 x \tan^2 x = 1 + 3\sec^2 x \tan^2 x$

10. $\frac{\sin(x+y)}{\sin x \cos y} = \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y} = 1 + \frac{\cos x}{\sin x}\frac{\sin y}{\cos y} = 1 + \tan y \cot x$

11. $\cos(x+y)\cos y+\sin(x+y)\sin y$

$=(\cos x\cos y-\sin x\sin y)\cos y+(\sin x\cos y+\cos x\sin y)\sin y$

$=\cos x\cos^2 y-\sin x\sin y\cos y+\sin x\cos y\sin y+\cos x\sin^2 y$

$=\cos x(\cos^2 y+\sin^2 y)=\cos x$

12. $\dfrac{\sin(x-y)}{\cos y}=\dfrac{\sin x\cos y-\cos x\sin y}{\cos y}=\sin x-\cos x\dfrac{\sin y}{\cos y}=\sin x-\cos x\tan y$

13. $\cos(\frac{3\pi}{4}+x)+\sin(\frac{3\pi}{4}-x)=(\cos\frac{3\pi}{4}\cos x-\sin\frac{3\pi}{4}\sin x)+(\sin\frac{3\pi}{4}\cos x-\cos\frac{3\pi}{4}\sin x)$

$=-\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x-(-\frac{1}{\sqrt{2}}\sin x)=0$

14. $\dfrac{\tan(\frac{\pi}{4}+x)-\tan(\frac{\pi}{4}-x)}{\tan(\frac{\pi}{4}+x)+\tan(\frac{\pi}{4}-x)}=\dfrac{\frac{1+\tan x}{1-\tan x}-\frac{1-\tan x}{1+\tan x}}{\frac{1+\tan x}{1-\tan x}+\frac{1-\tan x}{1+\tan x}}$

$=\dfrac{(1+\tan x)^2-(1-\tan x)^2}{(1+\tan x)^2+(1-\tan x)^2}=\dfrac{4\tan x}{2+2\tan^2 x}=\dfrac{2\tan x}{1+\tan^2 x}=\dfrac{2\frac{\sin x}{\cos x}}{\sec^2 x}=2\dfrac{\sin x}{\cos x}\cos^2 x=2\sin x\cos x$

15. $\sin(x+y)\sin(x-y)=(\sin x\cos y+\cos x\sin y)(\sin x\cos y-\cos x\sin y)$

$=\sin^2 x\cos^2 y-\cos^2 x\sin^2 y=(1-\cos^2 x)\cos^2 y-\cos^2 x(1-\cos^2 y)=\cos^2 y-\cos^2 x$

16. $\tan(x+y)\tan(x-y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}\times\dfrac{\tan x-\tan y}{1+\tan x\tan y}=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

$=\dfrac{\frac{\sin^2 x}{\cos^2 x}-\frac{\sin^2 y}{\cos^2 y}}{1-\frac{\sin^2 x}{\cos^2 x}\times\frac{\sin^2 y}{\cos^2 y}}=\dfrac{\sin^2 x\cos^2 y-\cos^2 x\sin^2 y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$

$=\dfrac{\sin^2 x(1-\sin^2 y)-\sin^2 y(1-\sin^2 x)}{\cos^2 x(1-\sin^2 y)-\sin^2 y(1-\cos^2 x)}=\dfrac{\sin^2 x-\sin^2 y}{\cos^2 x-\sin^2 y}$

Exercise 6.4

17. $\tan x=\tan[(x-y)+y]=\dfrac{\tan(x-y)+\tan y}{1-\tan(x-y)\tan y}$

18. $\sin 5x=\sin(2x+3x)=\sin 2x\cos 3x+\cos 2x\sin 3x$

$=\sin 2x\cos(x+2x)+\cos 2x\sin(x+2x)$

$=\sin 2x(\cos x\cos 2x-\sin x\sin 2x)+\cos 2x(\sin x\cos 2x+\cos x\sin 2x)$

$=\sin x(\cos^2 2x-\sin^2 2x)+2\cos x\cos 2x\sin 2x$

19. $\sin(\frac{\pi}{2}-x)\cot(\frac{\pi}{2}+x)=\cos x(-\tan x)=\cos x\frac{-\sin x}{\cos x}=-\sin x$

20. $\cos(-x)+\cos(\pi-x)=\cos x-\cos x=0$

and $\cos(\pi+x)+\cos x=-\cos x+\cos x=0$

21. $\dfrac{\sin(\pi-x)}{\tan(\pi+x)}\ \dfrac{\cot(\frac{\pi}{2}-x)}{\tan(\frac{\pi}{2}+x)}\ \dfrac{\cos(2\pi-x)}{\sin(-x)}=\dfrac{\sin x}{\tan x}\ \dfrac{\tan x}{-\cot x}\ \dfrac{\cos x}{-\sin x}=\dfrac{\cos x}{\cot x}=\sin x$

22. $\dfrac{\sin(-x)}{\sin(\pi+x)}-\dfrac{\tan(\frac{\pi}{2}+x)}{\cot x}+\dfrac{\cos x}{\sin(\frac{\pi}{2}+x)}=\dfrac{-\sin x}{-\sin x}-\dfrac{-\cot x}{\cot x}+\dfrac{\cos x}{\cos x}=1+1+1=3$

23. $\dfrac{\csc(\pi-x)}{\sec(\pi+x)}\ \dfrac{\cos(-x)}{\cos(\frac{\pi}{2}+x)}=\dfrac{\csc x}{-\sec x}\ \dfrac{\cos x}{-\sin x}=\dfrac{-\cos x}{\sin x}\ \dfrac{\cos x}{-\sin x}=\dfrac{\cos^2 x}{\sin^2 x}=\cot^2 x$

24. $\dfrac{\cos(\frac{\pi}{2}+x)\sec(-x)\tan(\pi-x)}{\sec(2\pi+x)\sin(\pi+x)\cot(\frac{\pi}{2}-x)}=\dfrac{-\sin x\sec x(-\tan x)}{\sec x(-\sin x)\tan x}=-1$

25. $\dfrac{\sin(\pi-x)\cos(\pi+x)\tan(2\pi-x)}{\sec(\frac{\pi}{2}+x)\csc(\frac{3\pi}{2}-x)\cot(\frac{3\pi}{2}+x)}=\dfrac{\sin x(-\cos x)(-\tan x)}{-\csc x(-\sec x)(-\tan x)}$

$=\dfrac{\sin x\cos x}{\dfrac{1}{-\sin x\cos x}}=-\sin^2 x\cos^2 x=-\sin^2 x(1-\sin^2 x)=-\sin^2 x+\sin^4 x$

Exercise 6.4

26. $\frac{\sin 2x}{1+\cos 2x}=\frac{2\sin x\cos x}{1+2\cos^2 x-1}=\frac{2\sin x\cos x}{2\cos^2 x}=\frac{\sin x}{\cos x}=\tan x$

27. $\frac{1+\cos x}{\sin x}=\frac{1+2\cos^2\frac{x}{2}-1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\cot\frac{x}{2}$

28. $2\csc 2x=\frac{2}{\sin 2x}=\frac{2}{2\sin x\cos x}=\frac{1}{\sin x}\frac{1}{\cos x}=\csc x\sec x$

29. $2\cot 2x=\frac{2}{\tan 2x}=\frac{2}{\frac{2\tan x}{1-\tan^2 x}}=\frac{1-\tan^2 x}{\tan x}=\frac{1}{\tan x}-\tan x=\cot x-\tan x$

30. $\frac{\cos 2x}{1+\sin 2x}=\frac{\cos^2 x-\sin^2 x}{1+2\sin x\cos x}=\frac{\cos^2 x-\sin^2 x}{\sin^2 x+2\sin x\cos x+\cos^2 x}$

$=\frac{(\cos x-\sin x)(\cos x+\sin x)}{(\sin x+\cos x)^2}$

$=\frac{\cos x-\sin x}{\cos x+\sin x}$ and $\tan(\frac{\pi}{4}-x)=\frac{1-\tan x}{1+(1)\tan x}=\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}=\frac{\cos x-\sin x}{\cos x+\sin x}$

31. $\sec 2x-\tan 2x=\frac{1-\sin 2x}{\cos 2x}=\frac{\sin^2 x-2\sin x\cos x+\cos^2 x}{\cos^2 x-\sin^2 x}$

$=\frac{(\cos x-\sin x)^2}{(\cos x-\sin x)(\cos x+\sin x)}=\frac{\cos x-\sin x}{\cos x+\sin x}$

32. $\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=\frac{1-(1-2\sin^2 x)+2\sin x\cos x}{1+(2\cos^2 x-1)+2\sin x\cos x}=\frac{2\sin x(\cos x+\sin x)}{2\cos x(\cos x+\sin x)}=\tan x$

33. $\cos 2x(1-\frac{1}{4}\sin^2 2x)=(1-2\sin^2 x)(1-\sin^2 x\cos^2 x)$

$=1-\sin^2 x\cos^2 x-2\sin^2 x+2\sin^4 x\cos^2 x$

$=1-\sin^2 x(1-\sin^2 x)-2\sin^2 x+2\sin^4 x(1-\sin^2 x)$

$=1-\sin^2 x+\sin^4 x-2\sin^2 x+2\sin^4 x-2\sin^6 x$

$=(1-3\sin^2 x+3\sin^4 x-\sin^6 x)-\sin^6 x$

$=(1-\sin^2 x)^3-\sin^6 x=(\cos^2 x)^3-\sin^6 x=\cos^6 x-\sin^6 x$

34. $4(\cos^6x+\sin^6x)=4(\cos^2x+\sin^2x)(\cos^4x-\cos^2x\sin^2x+\sin^4x)$
$=4(1)[(\cos^4x+2\cos^2x\sin^2x+\sin^4x)-3\cos^2x\sin^2x]$
$=4[(\cos^2x+\sin^2x)^2-\frac{3}{4}\sin^22x]=4(1)^2-3(1-\cos^22x)$
$=4-3+3\cos^22x=1+3\cos^22x$

35. $\sec x-\tan x=\dfrac{1-\sin x}{\cos x}=\dfrac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}=\dfrac{\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$

$=\dfrac{(\sin\frac{x}{2}-\cos\frac{x}{2})^2}{(\cos\frac{x}{2}-\sin\frac{x}{2})(\cos\frac{x}{2}+\sin\frac{x}{2})}=\dfrac{-\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}}=\dfrac{-\tan\frac{x}{2}+1}{1+\tan\frac{x}{2}}$ (division by $\cos\frac{x}{2}$)

and $\tan(\frac{\pi}{4}-\frac{x}{2})=\dfrac{\tan\frac{\pi}{4}-\tan\frac{x}{2}}{1+\tan\frac{\pi}{4}\tan\frac{x}{2}}=\dfrac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}$

36. $\dfrac{\sin 2x}{1+\cos 2x}\ \dfrac{\cos x}{1+\cos x}=\dfrac{2\sin x\cos x}{1+2\cos^2x-1}\ \dfrac{\cos x}{1+\cos x}=\dfrac{\sin x}{1+\cos x}=\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}=\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\tan\frac{x}{2}$

37. $\sin^2x+\cos^4x=\sin^2x+(1-\sin^2x)^2=\sin^2x+1-2\sin^2x+\sin^4x=(1-\sin^2x)+\sin^4x$
$=\cos^2x+\sin^4x$

38. $(\tan x-1)(\cot x+1)=1+\tan x-\cot x-1=\tan x-\cot x$

39. $\sin x\tan^2x\cot^3x=\sin x\tan^2x\cot^2x\cot x=\sin x\dfrac{\cos x}{\sin x}=\cos x$

40. $(\sin x+\cos x)(\tan x+\cot x)=\dfrac{\sin^2x}{\cos x}+\cos x+\sin x+\dfrac{\cos^2x}{\sin x}$

$=\dfrac{\sin^2x+\cos^2x}{\cos x}+\dfrac{\sin^2x+\cos^2x}{\sin x}=\dfrac{1}{\cos x}+\dfrac{1}{\sin x}=\sec x+\csc x$

41. $\sin^2x(\csc^2x-2\cos^2x)=1-2\sin^2x\cos^2x=1-2\sin^2x(1-\sin^2x)=1-2\sin^2x+2\sin^4x$
$=(1-2\sin^2x+\sin^4x)+\sin^4x=(1-\sin^2x)^2+\sin^4x=\cos^4x+\sin^4x$

Exercise 6.4

42. $\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)$

$= (\sin x + \cos x)(1 - \sin x \cos x)$

43. $\cos(\frac{\pi}{12} - x)\sec\frac{\pi}{12} - \sin(\frac{\pi}{12} - x)\csc\frac{\pi}{12}$

$$= \frac{\cos\frac{\pi}{12}\cos x + \sin\frac{\pi}{12}\sin x}{\cos\frac{\pi}{12}} - \frac{\sin\frac{\pi}{12}\cos x - \cos\frac{\pi}{12}\sin x}{\sin\frac{\pi}{12}}$$

$$= \frac{\sin\frac{\pi}{12}\cos\frac{\pi}{12}\cos x + \sin^2\frac{\pi}{12}\sin x - \cos\frac{\pi}{12}\sin\frac{\pi}{12}\cos x + \cos^2\frac{\pi}{12}\sin x}{\cos\frac{\pi}{12}\sin\frac{\pi}{12}}$$

$$= \frac{\sin x(\sin^2\frac{\pi}{12} + \cos^2\frac{\pi}{12})}{\frac{1}{2}\sin\frac{\pi}{6}} = \frac{\sin x}{\frac{1}{4}} = 4\sin x$$

44. $\tan(x - y) + \tan(y - z) = \frac{\tan x - \tan y}{1 + \tan x \tan y} + \frac{\tan y - \tan z}{1 + \tan y \tan z}$

$$= \frac{\tan x + \tan x \tan y \tan z - \tan y - \tan^2 y \tan z + \tan y + \tan^2 y \tan x - \tan z - \tan x \tan y \tan z}{(1 + \tan x \tan y)(1 + \tan y \tan z)}$$

$$= \frac{(\tan x - \tan z) + \tan^2 y(\tan x - \tan z)}{(1 + \tan x \tan y)(1 + \tan y \tan z)} = \frac{\sec^2 y(\tan x - \tan z)}{(1 + \tan x \tan y)(1 + \tan y \tan z)}$$

45. $\sin 8x = 2\sin 4x \cos 4x = 4\sin 2x \cos 2x \cos 4x = 8\sin x \cos x \cos 2x \cos 4x$

46. $1 - 2\sin^2(\frac{\pi}{4} - \frac{x}{2}) = \cos 2(\frac{\pi}{4} - \frac{x}{2}) = \cos(\frac{\pi}{2} - x) = \sin x$

47. $\sin(x + y) + \sin(x - y) = \sin x \cos y + \cos x \sin y + \sin x \cos y - \cos x \sin y$

$= 2\sin x \cos y$

Exercise 6.4

48. $\frac{\sin(x-y)}{\sin x \sin y}+\frac{\sin(y-z)}{\sin y \sin z}+\frac{\sin(z-x)}{\sin z \sin x}$

$=\frac{\sin z \sin(x-y)+\sin x \sin(y-z)+\sin y \sin(z-x)}{\sin x \sin y \sin z}$

$$=\frac{\sin z \sin x \cos y-\sin z \cos x \sin y+\sin x \sin y \cos z-\sin x \cos y \sin z+\sin y \sin z \cos x-\sin y \cos z \sin x}{\sin x \sin y \sin z}$$

$=0$

49. $\tan x+\tan(\pi-x)+\cot(\frac{\pi}{2}+x)=\tan x-\tan x-\tan x=-\tan x$

and $\tan(2\pi-x)=-\tan x$

50. $\sin(\frac{\pi}{2}+x)\cos(\pi-x)\cot(\frac{3\pi}{2}+x)=\cos x(-\cos x)(-\tan x)$

and $\sin(\frac{\pi}{2}-x)\sin(\frac{3\pi}{2}-x)\cot(\frac{\pi}{2}+x)=\cos x(-\cos x)(-\tan x)$

51. $\tan(\frac{\pi}{2}-x)-\cot(\frac{3\pi}{2}-x)+\tan(2\pi-x)-\cot(\pi-x)$

$=\cot x-\tan x-\tan x+\cot x$

$=2(\cot x-\tan x)=2\frac{1-\tan^2 x}{\tan x}=2\frac{2-\sec^2 x}{\tan x}=\frac{4-2\sec^2 x}{\tan x}$

52. $\tan(x+y+z)=\tan[(x+y)+z]=\frac{\tan(x+y)+\tan z}{1-\tan(x+y)\tan z}=\frac{\frac{\tan x+\tan y}{1-\tan x \tan y}+\tan z}{1-\frac{\tan x+\tan y}{1-\tan x \tan y}\tan z}$

$$=\frac{\frac{\tan x+\tan y+\tan z-\tan x \tan y \tan z}{1-\tan x \tan y}}{\frac{1-\tan x \tan y-\tan x \tan z-\tan y \tan z}{1-\tan x \tan y}}$$

$$=\frac{\tan x+\tan y+\tan z-\tan x \tan y \tan z}{1-\tan x \tan y-\tan x \tan z-\tan y \tan z}$$

Exercise 6.4

53. $\csc^2(\frac{\pi}{2}-x)=\sec^2x$

and $1+\sin^2x\csc^2(\frac{\pi}{2}-x)=1+\sin^2x\sec^2x=1+\frac{\sin^2x}{\cos^2x}=1+\tan^2x=\sec^2x$

54. $\tan(\frac{\pi}{4}+x)+\tan(\frac{\pi}{4}-x)=\frac{1+\tan x}{1-\tan x}+\frac{1-\tan x}{1+\tan x}$

$$=\frac{1+2\tan x+\tan^2x+1-2\tan x+\tan^2x}{1-\tan^2x}$$

$$=\frac{2(1+\tan^2x)}{1-\tan^2x}=\frac{2\sec^2x}{1-\tan^2x}=\frac{\frac{2}{\cos^2x}}{\frac{\cos^2x-\sin^2x}{\cos^2x}}=\frac{2}{\cos 2x}=2\sec 2x$$

55. $\frac{1-\sin 2x}{\cos 2x}=\frac{1-2\sin x\cos x}{\cos^2x-\sin^2x}=\frac{\cos^2x-2\sin x\cos x+\sin^2x}{\cos^2x-\sin^2x}$

$$=\frac{(\cos x-\sin x)^2}{(\cos x-\sin x)(\cos x+\sin x)}$$

$$=\frac{\cos x-\sin x}{\cos x+\sin x}\times\frac{\cos x+\sin x}{\cos x+\sin x}=\frac{\cos^2x-\sin^2x}{\cos^2x+\sin^2x+2\sin x\cos x}=\frac{\cos 2x}{1+\sin 2x}$$

56. $\frac{\sin 4x}{1-\cos 4x}\times\frac{1-\cos 2x}{\cos 2x}=\frac{2\sin 2x\cos 2x}{2\sin^2 2x}\times\frac{1-\cos 2x}{\cos 2x}=\frac{1-\cos 2x}{\sin 2x}=\frac{2\sin^2x}{2\sin x\cos x}$

$$=\frac{\sin x}{\cos x}=\tan x$$

EXERCISE 6.5

1. (a) $x=\frac{\pi}{3}, \frac{2\pi}{3}$ (b) $x=\frac{\pi}{3}, \frac{5\pi}{3}$ (c) $x=\frac{3\pi}{4}, \frac{7\pi}{4}$

   (d) $x=\frac{2\pi}{3}, \frac{4\pi}{3}$ (e) $x=\frac{7\pi}{6}, \frac{11\pi}{6}$ (f) $x=\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

2. (a) $x=-\frac{3\pi}{4}$

   (b) $\tan^2 x=\tan x \Rightarrow \tan x(\tan x-1)=0 \Rightarrow \tan x=0$ and $x=-\pi, 0$
   or $\tan x=1$ and $x=-\frac{3\pi}{4}$

   (c) $\sin^2 x-\sin x-2=0 \Rightarrow (\sin x+1)(\sin x-2)=0 \Rightarrow \sin x=-1$ and $x=-\frac{\pi}{2}$

   (d) $\sin^2 x=\frac{3}{4} \Rightarrow \sin x=-\frac{\sqrt{3}}{2}$ and $x=-\frac{2\pi}{3}, -\frac{\pi}{3}$

   (e) $4\cos^2 x=3 \Rightarrow \cos^2 x=\frac{3}{4} \Rightarrow \cos x=\pm\frac{\sqrt{3}}{2}$ and $x=-\frac{5\pi}{6}, -\frac{\pi}{6}$

   (f) $2\csc x-1=\pm 3 \Rightarrow 2\csc x=-2 \Rightarrow \csc x=-1$ and $x=-\frac{\pi}{2}$

3. (a) $\sin x-\sin x\tan x=0 \Rightarrow \sin x(1-\tan x)=0 \Rightarrow \sin x=0$ and $x=0, \pi$
   or $\tan x=1$ and $x=\frac{\pi}{4}$

   (b) $\sin x\tan 3x=0 \Rightarrow \sin x=0$ and $x=-\pi, 0$ or $\tan 3x=0, 3x\in[-3\pi,0]$
   $\Rightarrow 3x=-3\pi, -2\pi, -\pi, 0$ and $x=-\pi, -\frac{2\pi}{3}, -\frac{\pi}{3}, 0$

   (c) $6\sin^2 x-5\cos x-2=0 \Rightarrow 6-6\cos^2 x-5\cos x-2=0 \Rightarrow 6\cos^2 x+5\cos x-4=0$
   $\Rightarrow (3\cos x+4)(2\cos x-1)=0 \Rightarrow \cos x=\frac{1}{2}$ and $x=\frac{\pi}{3}, \frac{5\pi}{3}$

   (d) $\sqrt{2}\sin x+\tan x=0 \Rightarrow \sqrt{2}\sin x+\frac{\sin x}{\cos x}=0$
   $\Rightarrow \sqrt{2}\sin x\cos x+\sin x=0$ iff $\cos x\neq 0$
   $\Rightarrow \sin x(\sqrt{2}\cos x+1)=0 \Rightarrow \sin x=0$ and $x=-\pi, 0, \pi$ or $\sqrt{2}\cos x+1=0$
   $\Rightarrow \cos x=-\frac{1}{\sqrt{2}}$ and $x=-\frac{3\pi}{4}, \frac{3\pi}{4}$

   (e) $\cos^2 x-3\sin^2 x=1 \Rightarrow 1-\sin^2 x-3\sin^2 x=1$
   $\Rightarrow \sin x=0$ and $x=-2\pi, -\pi, 0, \pi, 2\pi$

(f) $2\tan x = \sec x \Rightarrow \frac{2\sin x}{\cos x} = \frac{1}{\cos x} \Rightarrow \sin x = \frac{1}{2}$ iff $\cos x \neq 0$ and $x = -\frac{11\pi}{6}, -\frac{7\pi}{6}$

4. (a) $\cos 2x = \cos^2 x \Rightarrow 2\cos^2 x - 1 = \cos^2 x \Rightarrow \cos^2 x = 1$

$\Rightarrow \cos x = \pm 1$ and $x = -\pi, 0,$ or π

(b) $\sin 2x = \cos x \rightarrow 2\sin x \cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0, \ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Rightarrow \cos x = 0$ and $x = -\frac{\pi}{2}, \frac{\pi}{2}$ or $\sin x = \frac{1}{2}$ and $x = \frac{\pi}{6}$

(c) $\cos^2 x - 2\sin x \cos x - \sin^2 x = 0 \Rightarrow \cos 2x - \sin 2x = 0$

$\Rightarrow \sin 2x = \cos 2x \Rightarrow \tan 2x = 1 \Rightarrow 2x = \frac{\pi}{4}$ and $x = \frac{\pi}{8}$

(d) $\tan 2x = 8\cos^2 x - \cot x \Rightarrow \frac{2\tan x}{1 - \tan^2 x} = 8\cos^2 x - \cot x$

$\Rightarrow 2\tan x = 8\cos^2 x - \cot x - 8\sin^2 x + \tan x$

$\Rightarrow 0 = 8\cos^2 x - 8\sin^2 x - \cot x - \tan x$

$\Rightarrow 0 = 8\cos^2 x - 8\sin^2 x - \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \Rightarrow 0 = 8(\cos^2 x - \sin^2 x) - \frac{1}{\cos x \sin x}$

$\Rightarrow 0 = 4(2\sin x \cos x)(\cos^2 x - \sin^2 x) - 1$

$\Rightarrow 1 = 4\sin 2x \cos 2x \Rightarrow 1 = 2(2\sin 2x \cos 2x)$

$\Rightarrow \sin 4x = \frac{1}{2}, \ 0 \leq 4x \leq 2\pi \Rightarrow 4x = \frac{\pi}{6}, \frac{5\pi}{6}$ and $x = \frac{\pi}{24}, \frac{5\pi}{24}$

(e) $\tan x + \sec 2x = 1 \Rightarrow \frac{\sin x}{\cos x} + \frac{1}{\cos 2x} = 1 \Rightarrow \sin x \cos 2x + \cos x = \cos x \cos 2x$

$\Rightarrow \sin x(1 - 2\sin^2 x) + \cos x - \cos x(1 - 2\sin^2 x) = 0$

$\Rightarrow \sin x - 2\sin^3 x + \cos x - \cos x + 2\sin^2 \cos x = 0$

$\Rightarrow \sin x(1 - 2\sin^2 x + 2\sin x \cos x) = 0$

$\Rightarrow \sin x(\cos 2x + \sin 2x) = 0 \Rightarrow \sin x = 0$ and $x = 0$ or $\cos 2x = -\sin 2x$

$\Rightarrow \tan 2x = -1,$

$-\pi \leq 2x \leq \pi \Rightarrow 2x = -\frac{\pi}{4}, \frac{3\pi}{4}$ and $x = -\frac{\pi}{8}, \frac{3\pi}{8}$

(f) $2(\sin^4 x + \cos^4 x) = 1 \Rightarrow 2[(\sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x) - 2\sin^2 x \cos^2 x] = 1$

$\Rightarrow 2(\sin^2 x + \cos^2 x)^2 - 4\sin^2 x \cos^2 x = 1 \Rightarrow 2 - \sin^2 2x = 1 \Rightarrow \sin^2 2x = 1$

$\Rightarrow \sin 2x = \pm 1, \ -2\pi \leq 2x \leq 2\pi$

$\Rightarrow 2x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ and $x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

5. $2\tan x\cos^2 x+\sin x\tan x-2\tan x=0 \Rightarrow 2\sin x\cos x+\frac{\sin^2 x}{\cos x}-\frac{2\sin x}{\cos x}=0$

$\Rightarrow 2\sin x(1-\sin^2 x)+\sin^2 x-2\sin x=0 \Rightarrow -2\sin^3 x+\sin^2 x=0$

$\Rightarrow \sin^2 x(1-2\sin x)=0$

$\Rightarrow \sin x=0$ and $x=-2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi$ or $\sin x=\frac{1}{2}$ and $x=-\frac{11\pi}{6},\ -\frac{7\pi}{6},\ \frac{\pi}{6},\ \frac{5\pi}{6}$

6. $\tan 2x=-\frac{24}{7} \Rightarrow \frac{2\tan x}{1-\tan^2 x}=-\frac{24}{7}$

$\Rightarrow 24\tan^2 x-14\tan x-24=0 \Rightarrow 12\tan^2 x-7\tan x-12=0$

$\Rightarrow (4\tan x+3)(3\tan x-4)=0 \Rightarrow \tan x=-\frac{3}{4}$, $\sin x=\pm\frac{3}{5}$ and $\cos x=\mp\frac{4}{5}$
or $\tan x=\frac{4}{3}$, $\sin x=\pm\frac{4}{5}$ and $\cos x=\mp\frac{3}{5}$

7. (a) $\sin x=x \Rightarrow x=0$. $y=\sin x$ is below $y=x$ for $x>0$ and $y=\sin x$ is above $y=x$ for $x<0$. $y=x$ is tangent to $y=\sin x$ at $x=0$.

(b) $\cos x=x \Rightarrow x\doteq 0.75$. The approximation is obtained by graphing $y=\cos x$ and $y=x$ on the same set of axes and estimating the value of x at the point of intersection. Now $\cos(0.75)\doteq 0.732$.
We can improve our approximation by examining values close to 0.75 using a calculator. $\cos(0.739)=0.7391$.

(c) $\tan x=-x \Rightarrow x=0$. This is the only point of intersection of $y=\tan x$ and $y=-x$ in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$.

(d) $\cos x=-\frac{x}{3} \Rightarrow x\doteq -1.17$. The graphs of $y=\cos x$ and $y=-\frac{x}{3}$ meet close to $x=-1.2$. We then used a calculator to improve our approximation. If $x=-1.17$ $\cos(-1.17)\doteq 0.3902$ and $-\frac{-1.17}{3}\doteq 0.39$.

(e) $\sin x=x\sin x \Rightarrow \sin x-x\sin x=0 \Rightarrow \sin x(1-x)=0$
$\Rightarrow \sin x=0$ and $x=0$ or $x=1$.
Graphing $y=\sin x$ and $y=x\sin x$ on the same set of axes illustrates that there are only two points of intersection in the specified interval.

(f) $\tan x = 2x \Rightarrow x = 0$. The graphs indicate two points of intersection in the neighbourhood of $x = 1.2$ and $x = -1.2$. We use a calculator and get the improved approximations $x = 1.165$ and $x = -1.165$.

8. Squaring both equations we get $x^2\sin^2 A + 2xy\sin A\cos A + y^2\cos^2 A = p^2$ and $x^2\cos^2 A - 2xy\cos A\sin A + y^2\sin^2 A = q^2$. Adding the equations gives us $x^2(\sin^2 A + \cos^2 A) + y^2(\cos^2 A + \sin^2 A) = p^2 + q^2$. Therefore $x^2 + y^2 = p^2 + q^2$.

9. $p^2 = \sin^2 A + 2\sin A\cos A + \cos^2 A = 1 + 2\sin A\cos A \Rightarrow p^2 - 1 = 2\sin A\cos A$
$\Rightarrow q(p^2 - 1) = (\tan A + \cot A)2\sin A\cos A$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A\sin A} \times 2\sin A\cos A = 1 \times 2 = 2.$$

6.6 REVIEW EXERCISE

1. (a) $\sin(x+y)=\sin x\cos y+\cos x\sin y=\frac{4}{5}\times-\frac{12}{13}+\frac{3}{5}\times\frac{5}{13}=\frac{-48+15}{65}=-\frac{33}{65}$

(b) $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}=\frac{\frac{4}{3}+\frac{5}{12}}{1+\frac{4}{3}\times\frac{-5}{12}}=\frac{48+15}{36-20}=\frac{63}{16}$

(c) $\cos 2(x+y)=1-2\sin^2(x+y)=1-2\left(-\frac{33}{65}\right)^2=1-\frac{2178}{4225}=\frac{2047}{4225}$

2. (a) $\sin\frac{13\pi}{12}=-\sin\frac{\pi}{12}=-\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=-\left(\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}\right)$

$=-\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{-\sqrt{3}+1}{2\sqrt{2}}$

(b) $\cos\left(\frac{-11\pi}{12}\right)=\cos\left(\frac{11}{12}\pi\right)=-\cos\frac{\pi}{12}=-\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=-\left(\cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)$

$=-\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{-\sqrt{3}-1}{2\sqrt{2}}$

(c) $-\tan\left(-\frac{5\pi}{12}\right)=\tan\frac{5\pi}{12}=\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}}=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$

(d) $\sin 15^\circ=\sin(45-30)=\sin 45\cos 30-\cos 45\sin 30=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}$

(e) $\cos(-75^\circ)=\cos 75=\cos(45+30)=\cos 45\cos 30-\sin 45\sin 30$

$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}$

(f) $-\tan 105^\circ=-\tan(60+45)=-\frac{\tan 60+\tan 45}{1-\tan 60\tan 45}=-\frac{\sqrt{3}+1}{1-\sqrt{3}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$

6.6 Review Exercise

3. (a) $\sin 2x = 2\sin x\cos x = 2\times\frac{3}{5}\times\frac{-4}{5} = -\frac{24}{25}$

(b) $\cos 2x = 1-2\sin^2 x = 1-2\left(\frac{3}{5}\right)^2 = 1-\frac{18}{25} = \frac{7}{25}$

(c) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x} = \dfrac{2\times-\frac{3}{4}}{1-\frac{9}{16}} = \dfrac{-\frac{3}{2}}{\frac{7}{16}} = -\frac{3}{2}\times\frac{16}{7} = -\frac{24}{7}$

4. (a) $\cos x = 1-2\sin^2\frac{x}{2} = 1-\frac{8}{9} = \frac{1}{9}$

(b) $\tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{2\times\frac{2}{\sqrt{5}}}{1-\frac{4}{5}} = \frac{4}{\sqrt{5}}\times 5 = 4\sqrt{5}$

(c) $\cos\frac{x}{2} = 1-2\sin^2\frac{x}{4} \Rightarrow \frac{\sqrt{5}}{3} = 1-2\sin^2\frac{x}{4} \Rightarrow \sin^2\frac{x}{4} = \frac{1}{2}-\frac{\sqrt{5}}{6} \Rightarrow \sin\frac{x}{4} = \sqrt{\frac{3-\sqrt{5}}{6}}$

5. (a) $\cos 225^\circ = 1-2\sin^2 112\frac{1}{2}^\circ \Rightarrow -\frac{1}{\sqrt{2}} = 1-2\sin^2 112\frac{1}{2}^\circ \Rightarrow 2\sin^2 112\frac{1}{2}^\circ = 1+\frac{1}{\sqrt{2}}$

$\sin 112\frac{1}{2}^\circ = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

(b) $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8}-1 \Rightarrow 2\cos^2\frac{\pi}{8} = \frac{1}{\sqrt{2}}+1 \Rightarrow \cos\frac{\pi}{8} = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$

(c) $\tan\frac{3\pi}{4} = \dfrac{2\tan\frac{3\pi}{8}}{1-\tan^2\frac{3\pi}{8}} \Rightarrow -1 = \dfrac{2\tan\frac{3\pi}{8}}{1-\tan^2\frac{3\pi}{8}} \Rightarrow \tan^2\frac{3\pi}{8}-2\tan\frac{3\pi}{8}-1=0$

$\Rightarrow \tan\frac{3\pi}{8} = \dfrac{2+\sqrt{4+4}}{2} = 1+\sqrt{2}$. Now $\tan\frac{3\pi}{8} = \dfrac{2\tan\frac{3\pi}{16}}{1-\tan^2\frac{3\pi}{16}} \Rightarrow 1+\sqrt{2} = \dfrac{2\tan\frac{3\pi}{16}}{1-\tan^2\frac{3\pi}{16}}$

$\Rightarrow 0 = (1+\sqrt{2})\tan^2\frac{3\pi}{16}+2\tan\frac{3\pi}{16}-(1+\sqrt{2})$

$\Rightarrow \tan\frac{3\pi}{16} = \dfrac{-2+\sqrt{4+4(1+2\sqrt{2}+2)}}{2(1+\sqrt{2})} = \dfrac{-2+\sqrt{16+8\sqrt{2}}}{2(1+\sqrt{2})} = \dfrac{-2+2\sqrt{4+2\sqrt{2}}}{2(1+\sqrt{2})}$

$\Rightarrow \tan\frac{3\pi}{16} = \dfrac{-1+\sqrt{4+2\sqrt{2}}}{1+\sqrt{2}}$

6.6 Review Exercise

6\. (a) $\sin 120^\circ = \sin(180 - 60) = \sin 60 = \frac{\sqrt{3}}{2}$

(b) $\cos \frac{11\pi}{6} = \cos(2\pi - \frac{\pi}{6}) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(c) $\tan\left(-\frac{7}{3}\pi\right) = -\tan(2\pi + \frac{\pi}{3}) = -\tan \frac{\pi}{3} = -\sqrt{3}$

7\. (a) $\sin\left(-\frac{7\pi}{6}\right) = -\sin \frac{7\pi}{6} = -\sin(\frac{3\pi}{2} - \frac{\pi}{3}) = \cos \frac{\pi}{3} = \frac{1}{2}$

(b) $\cos 495^\circ = \cos 135 = \cos(90 + 45) = -\sin 45 = -\frac{1}{\sqrt{2}}$

(c) $\tan \frac{39\pi}{4} = \tan \frac{7\pi}{4} = \tan(\frac{3\pi}{2} + \frac{\pi}{4}) = -\cot \frac{\pi}{4} = -1$

8\. (a) $\csc 2x - \cot 2x = \frac{1}{2\sin x \cos x} - \frac{\cos^2 x - \sin^2 x}{2\sin x \cos x} = \frac{(1 - \cos^2 x) + \sin^2 x}{2\sin x \cos x}$

$= \frac{2\sin^2 x}{2\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$

(b) $\frac{1 - \sin 2x}{\cos 2x} = \frac{\cos^2 x - 2\sin x \cos x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{(\cos x - \sin x)^2}{(\cos x - \sin x)(\cos x + \sin x)}$

$= \frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}} = \frac{1 - \tan x}{1 + \tan x}$

(c) $\sec y \cos(x + y) = \sec y(\cos x \cos y - \sin x \sin y) = \cos x - \sin x \frac{\sin y}{\cos y}$

$= \cos x - \sin x \tan y$

(d) $\sin(\pi + x) + \cos\left(\frac{\pi}{2} - x\right) + \tan\left(\frac{\pi}{2} + x\right) = -\sin x + \sin x - \cot x = -\cot x$

(e) $\frac{\sin 4x - \sin 2x}{\sin 2x} = \frac{2\sin 2x \cos 2x - \sin 2x}{\sin 2x} = 2\cos 2x - 1$

and $\frac{\cos 3x}{\cos x} = \frac{\cos(2x + x)}{\cos x} = \frac{\cos 2x \cos x - \sin 2x \sin x}{\cos x} = \cos 2x - \frac{2\sin^2 x \cos x}{\cos x}$

$= \cos 2x - 2\sin^2 x = \cos 2x - (1 - \cos 2x) = 2\cos 2x - 1$

(f) $\cos x+\cos 2x+\cos 3x=\cos x+\cos 2x+\cos(2x+x)$

$=\cos x+\cos 2x+\cos 2x\cos x-\sin 2x\sin x$

$=\cos x+\cos 2x+\cos 2x\cos x-2\sin^2 x\cos x$

$=\cos x+\cos 2x+\cos 2x\cos x-2\cos x(1-\cos^2 x)$

$=\cos 2x(1+\cos x)+\cos x(1-2+2\cos^2 x)=\cos 2x(1+\cos x)+\cos x(2\cos^2 x-1)$

$=\cos 2x(1+\cos x)+\cos x\cos 2x=\cos 2x(1+2\cos x)$

(g) $\sin(x+y)+\sin(x-y)=\sin x\cos y+\cos x\sin y+(\sin x\cos y-\cos x\sin y)$

$=2\sin x\cos y$

9. (a) $2\sin x\cos x=0 \Rightarrow \sin x=0$ and $x=0, \pi$ or $\cos x=0$ and $x=\frac{\pi}{2}$

(b) $\sin^2 x+\sin x=0 \Rightarrow \sin x(\sin x+1)=0$

$\Rightarrow \sin x=0$ and $x=-\pi, 0, \pi$ or $\sin x=-1$ and $x=-\frac{\pi}{2}$

(c) $\cos^2 x-\cos x=0 \Rightarrow \cos x(\cos x-1)=0$

$\Rightarrow \cos x=0$ and $x=\frac{\pi}{2}, \frac{3\pi}{2}$ or $\cos x=1$ and $x=0, 2\pi$

(d) $\sin^2 x-2\sin x+1=0 \Rightarrow (\sin x-1)^2=0 \Rightarrow \sin x=1$ and $x=-\frac{3\pi}{2}, \frac{\pi}{2}$

(e) $\cos^2 2x+2\cos 2x+1=0 \Rightarrow (\cos 2x+1)^2=0 \Rightarrow \cos 2x=-1,\ -2\pi\le 2x\le 2\pi$

$\Rightarrow 2x=-\pi, \pi \Rightarrow x=-\frac{\pi}{2}, \frac{\pi}{2}$

(f) $\sec^2 2x-1=0 \Rightarrow \sec 2x=1,\ -4\pi\le 2x\le 4\pi$ and $2x=-4\pi, -2\pi, 0, 2\pi, 4\pi$

$\Rightarrow x=-2\pi, -\pi, 0, \pi, 2\pi$ or $\sec 2x=-1$ and $2x=-3\pi, -\pi, \pi, 3\pi$

$\Rightarrow x=-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

(g) $\tan 4x-\tan 2x=0 \Rightarrow \frac{2\tan 2x}{1-\tan^2 2x}-\tan 2x=0 \Rightarrow \tan 2x\left(\frac{2}{1-\tan^2 2x}-1\right)=0$

$\Rightarrow \tan 2x\left(\frac{1+\tan^2 2x}{1-\tan^2 2x}\right)=0 \Rightarrow \tan 2x(\sec^2 2x)=0,\ 0\le 2x\le 2\pi,\ \tan 2x\ne\pm 1$

$\Rightarrow \tan 2x=0$ and $2x=0, \pi, 2\pi \Rightarrow x=0, \frac{\pi}{2}, \pi$

(h) $\sqrt{3}\cos x+\sin x=0 \Rightarrow \sin x=-\sqrt{3}\cos x \Rightarrow \tan x=-\sqrt{3} \Rightarrow x=-\frac{4\pi}{3}, -\frac{\pi}{3}$

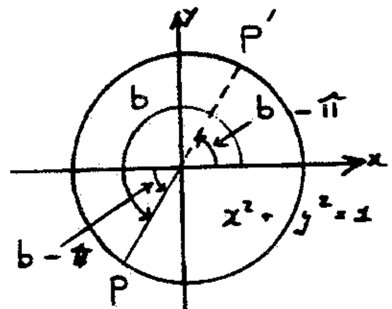
6.6 Review Exercise

10. (a) $\csc(x+y)=\dfrac{1}{\sin x\cos y+\cos x\sin y}=\dfrac{1}{\frac{1}{\sqrt{2}}\times\frac{7}{25}+\frac{1}{\sqrt{2}}\times\frac{24}{25}}=\dfrac{25\sqrt{2}}{31}$

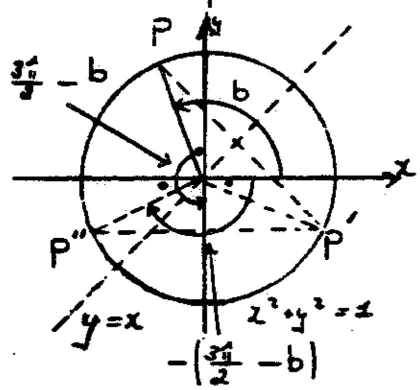
(b) $\sec(x-y)=\dfrac{1}{\cos x\cos y+\sin x\sin y}=\dfrac{1}{\frac{1}{\sqrt{2}}\times\frac{7}{25}+\frac{1}{\sqrt{2}}\times\frac{24}{25}}=\dfrac{25\sqrt{2}}{31}$

11. $\cos 12a=2\cos^2 6a-1=2(2\cos^2 3a-1)^2-1=8\cos^4 3a-8\cos^2 3a+1;$
$\cos 12a=1-2\sin^2 6a=1-2(2\sin 3a\cos 3a)^2=1-8\sin^2 3a(1-\sin^2 3a)$
$=1-8\sin^2 3a+8\sin^4 3a$

12. (a) b determines the point $P(\cos b,\sin b)$ on the unit circle. $P'(-\cos b,-\sin b)$ is the image of P after a reflection in the origin. But P′ is determined by $b-\pi$ and has coordinate $P'\left[\cos(b-\pi),\ \sin(b-\pi)\right]$.
Therefore $\sin(b-\pi)=-\sin b$.

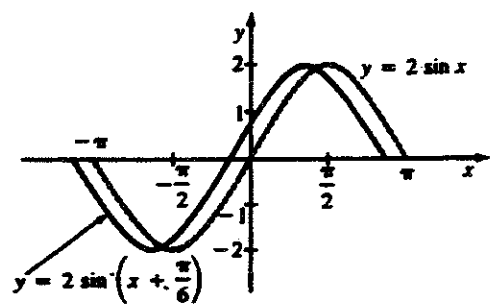


(b) b determines $P(\cos b,\sin b)$ on the unit circle. $P'(\sin b,\cos b)$ is the image of P after a reflection in the line $y=x$. $P''(-\sin b,\cos b)$ is the image of P″ after a reflection in the y-axis. But P″ is determined by $b-\frac{3}{2}\pi$ and has coordinate $P''\left[\cos(b-\frac{3}{2}\pi),\sin(b-\frac{3}{2}\pi)\right]$.
Therefore $\cos(b-\frac{3}{2}\pi)=-\sin b$.



13. Let $\sqrt{3}\sin x+\cos x=a\sin(x+b)\Rightarrow\sqrt{3}\sin x+\cos x=a\sin x\cos b+a\cos x\sin b$
Comparing coefficients we get $a\cos b=\sqrt{3}$ and $a\sin b=1$. Since $a>0$, b is a first quadrant angle.
Divide the two equqtions: $\dfrac{a\sin b}{a\cos b}=\dfrac{1}{\sqrt{3}}\Rightarrow\tan b=\dfrac{1}{\sqrt{3}}\Rightarrow b=\dfrac{\pi}{6}$

Square both equations and add: $a^2\cos^2 b+a^2\sin^2 b=3+1\Rightarrow a^2(\cos^2 b+\sin^2 b)=4$
$\Rightarrow a^2=4\Rightarrow a=2$
Therefore $\sqrt{3}\sin x+\cos x=2\sin(x+\frac{\pi}{6})$



14. Let $\sqrt{3}\sin x-\cos x=a\cos(x+b)$ $\Rightarrow$ $\sqrt{3}\sin x-\cos x=a\cos x\cos b-a\sin x\sin b$

Comparing coefficients we get $a\cos b=-1$ and $a\sin b=-\sqrt{3}$.

Since $a>0$, b is a third quadrant angle.

Divide the two equqtions: $\frac{a\sin b}{a\cos b}=\sqrt{3}$ $\Rightarrow$ $\tan b=\sqrt{3}$ $\Rightarrow$ $b=\frac{4}{3}\pi$

Square both equations and add: $a^2\cos^2 b+a^2\sin^2 b=1+3$ $\Rightarrow$ $a=2$

Therefore $\sqrt{3}\sin x-\cos x=2\cos\left(x+\frac{4\pi}{3}\right)$

15. $4\sin a+3\sin\frac{b}{2}+2\cos\frac{c}{2}+\sin\frac{d}{2}=4\sin a+3\sin\frac{\pi-a}{2}+2\cos\frac{2\pi+a}{2}+\sin\frac{3\pi-a}{2}$

$=4\sin a+3\sin\left(\frac{\pi}{2}-\frac{a}{2}\right)+2\cos\left(\pi+\frac{a}{2}\right)+\sin\left(\frac{3\pi}{2}-\frac{a}{2}\right)$

$=4\sin a+3\cos\frac{a}{2}-2\cos\frac{a}{2}-\cos\frac{a}{2}=4\sin a=4k$

16 If $\theta=18^\circ$, $\sin 2\theta=\sin 36^\circ=\cos(90^\circ-36^\circ)=\cos 54^\circ=\cos 3\theta$

$\sin 36^\circ=\cos 54^\circ$ $\Rightarrow$ $\sin 2(18^\circ)=\cos 3(18^\circ)$ $\Rightarrow$ $2\sin 18^\circ\cos 18^\circ=4\cos^3 18^\circ-3\cos 18^\circ$

$\Rightarrow$ $2\sin 18^\circ=4\cos^2 18^\circ-3$

$\Rightarrow$ $2\sin 18^\circ=4(1-\sin^2 18^\circ)-3 \Rightarrow 4\sin^2 18^\circ+2\sin 18^\circ-1=0$

$\Rightarrow$ $\sin 18^\circ=\frac{-2+\sqrt{4+16}}{8}=\frac{-2+2\sqrt{5}}{8}=\frac{-1+\sqrt{5}}{4}$

$\cos 18^\circ=\sqrt{1-\sin^2 18^\circ}=\sqrt{1-\frac{1-2\sqrt{5}+5}{16}}=\sqrt{\frac{10+2\sqrt{5}}{16}}=\sqrt{\frac{5+\sqrt{5}}{8}}$

17. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(3\sin\theta-\sin 3\theta)^{\frac{2}{3}}+(\cos 3\theta+3\cos\theta)^{\frac{2}{3}}$

$=(3\sin\theta-3\sin\theta+4\sin^3\theta)^{\frac{2}{3}}+(4\cos^3\theta-3\cos\theta+3\cos\theta)^{\frac{2}{3}}$

$=4^{\frac{2}{3}}\sin^2\theta+4^{\frac{2}{3}}\cos^2\theta=4^{\frac{2}{3}}(\sin^2\theta+\cos^2\theta)=4^{\frac{2}{3}}$

18. $\sin(x+y)=a\sin(x-y) \Rightarrow \sin x\cos y+\cos x\sin y=a\sin x\cos y-a\cos x\sin y$

$\Rightarrow (1+a)\cos x\sin y=(a-1)\sin x\cos y$

$\Rightarrow (1+a)\cos x\tan y=(a-1)\sin x \Rightarrow \tan y=\dfrac{(a-1)\sin x}{(a+1)\cos x}$

$\cos(x+y)=b\cos(x-y) \Rightarrow \cos x\cos y-\sin x\sin y=b\cos x\cos y+b\sin x\sin y$

$\Rightarrow (1-b)\cos x\cos y=(1+b)\sin x\sin y$

$\Rightarrow (1-b)\cos x=(1+b)\sin x\tan y \Rightarrow \tan y=\dfrac{(1-b)\cos x}{(1+b)\sin x}$

Therefore $\dfrac{(a-1)\sin x}{(a+1)\cos x}=\dfrac{(1-b)\cos x}{(1+b)\sin x} \Rightarrow (a-1)(1+b)\sin^2 x=(a+1)(1-b)\cos^2 x$

$\Rightarrow (a-1)(1+b)\sin^2 x=(a+1)(1-b)(1-\sin^2 x)$

$\Rightarrow [(a-1)(1+b)+(a+1)(1-b)]\sin^2 x=(a+1)(1-b)$

$\Rightarrow 2(a-b)\sin^2 x=(a+1)(1-b) \Rightarrow \sin^2 x=\dfrac{(a+1)(1-b)}{2(a-b)}$

Now $\cos 2x=1-2\sin^2 x=1-\dfrac{(a+1)(1-b)}{(a-b)}=\dfrac{a-b-a+ab-1+b}{a-b}=\dfrac{ab-1}{a-b}$

6.7 CHAPTER 6 TEST

1. (a) $\cos(-\frac{\pi}{12})=\cos\frac{\pi}{12}=\cos(\frac{\pi}{4}-\frac{\pi}{6})=\cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}$

(b) $\cos\frac{3\pi}{4}=1-2\sin^2\frac{3\pi}{8} \Rightarrow -\frac{1}{\sqrt{2}}=1-2\sin^2\frac{3\pi}{8} \Rightarrow \sin^2\frac{3\pi}{8}=\frac{1}{2}+\frac{1}{2\sqrt{2}}$

$\Rightarrow \sin\frac{3\pi}{8}=\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

(c) $\frac{\tan 67^\circ-\tan 22^\circ}{1+\tan 67^\circ\tan 22^\circ}=\tan 45^\circ=1$

(d) $(\sin\frac{\pi}{8}+\cos\frac{\pi}{8})^2=\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}+2\sin\frac{\pi}{8}\cos\frac{\pi}{8}=1+\sin\frac{\pi}{4}=\frac{\sqrt{2}+1}{\sqrt{2}}$

(e) $\sin\frac{13}{36}\pi\cos\frac{5}{36}\pi+\cos\frac{13}{36}\pi\sin\frac{5}{36}\pi=\sin(\frac{13}{36}\pi+\frac{5}{36}\pi)=\sin\frac{\pi}{2}=1$

2. $\sin[2(x-y)]=2\sin(x-y)\cos(x-y)$

$=2(\sin x\cos y-\cos x\sin y)(\cos x\cos y+\sin x\sin y)$

$=2\left(\frac{12}{13}\times\frac{4}{5}-\frac{5}{13}\times\frac{-3}{5}\right)\left(\frac{5}{13}\times\frac{4}{5}+\frac{12}{13}\times\frac{-3}{5}\right)=2\left(\frac{48+15}{65}\right)\left(\frac{20-36}{65}\right)=\frac{-2016}{4225}$

3. $\tan 2x=\frac{2\tan x}{1-\tan^2 x}=\frac{2(-\frac{3}{4})}{1-(-\frac{3}{4})^2}=\frac{-\frac{3}{2}}{1-\frac{9}{16}}=-\frac{3}{2}\times\frac{16}{7}=-\frac{24}{7}$

4. $(\sqrt{2}\cos x-1)(\sqrt{2}\cos x+1)=\frac{1-3}{4} \Rightarrow 2\cos^2 x-1=-\frac{1}{2} \Rightarrow \cos 2x=-\frac{1}{2}$

$\cos 4x=2\cos^2 2x-1=2(\frac{1}{4})-1=-\frac{1}{2}$

5. (a) $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x}=\dfrac{1+2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}-1}{1+2\sin\frac{x}{2}\cos\frac{x}{2}-1+2\sin^2\frac{x}{2}}$

$=\dfrac{2\cos\frac{x}{2}(\sin\frac{x}{2}+\cos\frac{x}{2})}{2\sin\frac{x}{2}(\cos\frac{x}{2}+\sin\frac{x}{2})}=\cot\frac{x}{2}$

(b) $\cos(x+y)\cos(x-y)=(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)$

$=\cos^2x\cos^2y-\sin^2x\sin^2y=(1-\sin^2x)(1-\sin^2y)-\sin^2x\sin^2y$

$=1-\sin^2y-\sin^2x+\sin^2x\sin^2y-\sin^2x\sin^2y=1-(\sin^2x+\sin^2y)$

(c) $\dfrac{\sin(x-\pi)}{\cos(\pi+x)}-\dfrac{\cos(\frac{\pi}{2}-x)}{\sin(-\pi-x)}=\dfrac{-\sin(\pi-x)}{-\cos x}-\dfrac{\sin x}{-\sin(\pi+x)}=\dfrac{-\sin x}{-\cos x}+\dfrac{\sin x}{-\sin x}$

$=\dfrac{\sin x}{\cos x}-1=\dfrac{\sin x-\cos x}{\cos x}$

6. (a) $\tan^2(2x-\frac{\pi}{12})=3 \Rightarrow \tan(2x-\frac{\pi}{12})=\pm\sqrt{3},\quad -\frac{13}{12}\pi\le 2x-\frac{\pi}{12}\le\frac{11}{12}\pi$

$\Rightarrow 2x-\frac{\pi}{12}=-\frac{2\pi}{3},\ -\frac{\pi}{3},\ \frac{\pi}{3},\ \frac{2\pi}{3} \Rightarrow x=-\frac{7\pi}{24},\ -\frac{\pi}{8},\ \frac{5\pi}{24},\ \frac{3\pi}{8}$

(b) $\cos 2x-\cos^2x-2\sin x+3=0 \Rightarrow 1-2\sin^2x-1+\sin^2x-2\sin x+3=0$

$\Rightarrow \sin^2x+2\sin x-3=0 \Rightarrow (\sin x-1)(\sin x+3)=0 \Rightarrow \sin x=1 \Rightarrow x=\frac{\pi}{2}$

7. $\tan(a+b-c)=\tan[(a+b)-c]=\dfrac{\tan(a+b)-\tan c}{1+\tan(a+b)\tan c}=\dfrac{\dfrac{\tan a+\tan b}{1-\tan a\tan b}-\tan c}{1+\dfrac{\tan a+\tan b}{1-\tan a\tan b}\tan c}$

$=\dfrac{\tan a+\tan b-\tan c+\tan a\tan b\tan c}{1-\tan a\tan b+\tan a\tan c+\tan b\tan c}$