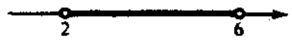
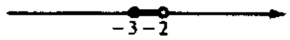
Review and Preview to Chapter 4

EXERCISE 1

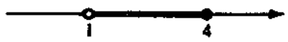
1. (a) $(-2,6)=\{x \mid -2<x<6\}$



(b) $[-3,-2)=\{x \mid -3\leq x<-2\}$

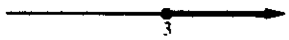


(c) $(1,4]=\{x \mid 1<x\leq 4\}$

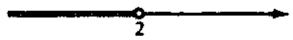


(d) $[-2,1.5]=\{x \mid -2\leq x\leq 1.5\}$

(e) $[3,\infty)=\{x \mid x\geq 3\}$



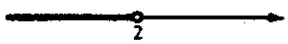
(f) $(-\infty,2)=\{x \mid x<2\}$



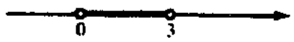
(g) $(-\infty,1]=\{x \mid x\leq 1\}$

(h) $(-\frac{3}{2},\infty)=\{x \mid x>-\frac{3}{2}\}$

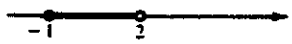
2. (a) $x<2 \Leftrightarrow (-\infty,2)$



(b) $0<x<3 \Leftrightarrow (0,3)$



(c) $-1\leq x<2 \Leftrightarrow [-1,2)$



(d) $x>1 \Leftrightarrow (1,\infty)$

(e) $-1\leq x\leq 3 \Leftrightarrow [-1,3]$

(f) $x\leq -1 \Leftrightarrow (-\infty,-1]$

EXERCISE 2

1. (a) $3x+7>0 \Rightarrow 3x>-7 \Rightarrow x>-\frac{7}{3} \Leftrightarrow x\in(-\frac{7}{3},\infty)$

(b) $18-4x<0 \Rightarrow 4x>18 \Rightarrow x>\frac{9}{2} \Leftrightarrow x\in(\frac{9}{2},\infty)$

(c) $17-2x\geq 13 \Rightarrow 2x\leq 4 \Rightarrow x\leq 2 \Leftrightarrow x\in(-\infty,2]$

(d) $2x+1<5x-11 \Rightarrow 3x>12 \Rightarrow x>4 \Leftrightarrow x\in(4,\infty)$

(e) $2x-1<19 \Rightarrow 2x<20 \Rightarrow x<10 \Leftrightarrow x\in(-\infty,10)$

(f) $x^2-7x+6>0 \Rightarrow (x-1)(x-6)>0 \Rightarrow x-1>0$ and $x-6>0$ or $x-1<0$ and

$x-6<0 \Rightarrow x>1$ and $x>6$ or $x<1$ and $x<6 \Rightarrow x>6$ or $x<1 \Leftrightarrow x \in (-\infty,1)\cup(6,\infty)$

(g) $12-x-x^2>0 \Rightarrow (x+4)(x-3)<0 \Rightarrow x+4<0$ and $x-3>0$ or $x+4>0$ and $x-3<0 \Rightarrow x<-4$ and $x>3$ or $x>-4$ and $x<3 \Rightarrow -4<x<3 \Leftrightarrow x \in (-4,3)$

(h) $x^2<3x \Rightarrow x^2-3x<0 \Rightarrow x(x-3)<0 \Rightarrow x<0$ and $x-3>0$ or $x>0$ and $x-3<0 \Rightarrow$ $x<0$ and $x>3$ or $x>0$ and $x<3 \Rightarrow 0<x<3 \Leftrightarrow x \in (0,3)$

(i) $x^2-9>0 \Rightarrow (x-3)(x+3)>0 \Rightarrow x-3>0$ and $x+3>0$ or $x-3<0$ and $x+3<0$ $\Rightarrow x>3$ and $x>-3$ or $x<3$ and $x<-3 \Rightarrow x>3$ or $x<-3 \Leftrightarrow x \in (-\infty,-3)\cup(3,\infty)$

(j) $x^2\le 5 \Leftrightarrow |x| \le \sqrt{5} \Leftrightarrow x\le\sqrt{5}$ and $x\ge -5 \Leftrightarrow x \in [-\sqrt{5},\sqrt{5}]$

(k) $(x+1)(2x+1)(x-6)>0$. For $(x+1)(2x+1)(x-6)=0$, the solutions are $x=-1$, $x=-\frac{1}{2}$, or $x=6$. This divides the real line into four intervals, $(-\infty,-1)$, $(-1,-\frac{1}{2})$, $(-\frac{1}{2},6)$, and $(6,\infty)$.

Interval	$x+1$	$2x+1$	$x-6$	$(x+1)(2x+1)(x-6)$
$x<-1$	$-$	$-$	$-$	$-$
$-1<x<-\frac{1}{2}$	$+$	$-$	$-$	$+$
$-\frac{1}{2}<x<6$	$+$	$+$	$-$	$-$
$x>6$	$+$	$+$	$+$	$+$

So the solution to $(x+1)(2x+1)(x-6)>0$ is $-1<x<-\frac{1}{2}$ or $x>6 \Leftrightarrow$ $x \in (-1,-\frac{1}{2})\cup(6,\infty)$

(l) $x^3+3x^2-10x<0 \Rightarrow x(x^2+3x-10) \Rightarrow x(x+5)(x-2)=0$, the solutions are $x=-5$, $x=0$, $x=2$. This divides the real line into four intervals $(-\infty,-5)$, $(-5,0)$, $(0,2)$ and $(2,\infty)$.

Interval	x	$x+5$	$x-2$	$x(x+5)(x-2)$
$x<-5$	$-$	$-$	$-$	$-$
$-5<x<0$	$-$	$+$	$-$	$+$
$0<x<2$	$+$	$+$	$-$	$-$
$x>2$	$+$	$+$	$+$	$+$

So the solution to $x^3+3x^2-10x<0$ is $x<-5$ or $0<x<2 \Leftrightarrow x \in (-\infty,-5)\cup(0,2)$

(m) $x^3+3x^2-4<0$; $x-1$ is a factor

$$\begin{array}{r|l} & x^2+4x+4 \\ x-1 & x^3+3x^2\quad -4 \\ & \underline{x^3-x^2} \\ & \quad 4x^2 \\ & \quad \underline{4x^2-4x} \\ & \qquad 4x-4 \\ & \qquad \underline{4x-4} \end{array}$$

Thus $x^3+3x^2-4=(x-1)(x^2+4x+4)$
$=(x-1)(x+2)^2$

So $(x-1)(x+2)^2<0 \Rightarrow x-1<0$ since $(x+2)^2$ is always positive $\Leftrightarrow x \in (-\infty,1)$.

(n) $x^3 + 2x^2 - 9x - 18 > 0 \Rightarrow x^2(x+2) - 9(x+2) > 0 \Rightarrow (x+2)(x+3)(x-3) > 0$.
For $(x+2)(x+3)(x-3) = 0$, the solutions are $x = -3$, $x = -2$, and $x = 3$. This divides the real line into four intervals $(-\infty, -3)$, $(-3, -2)$, $(-2, 3)$ and $(3, \infty)$.

Interval	$x+3$	$x+2$	$x-3$	$(x+3)(x+2)(x-3)$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < -2$	$+$	$-$	$-$	$+$
$-2 < x < 3$	$+$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$	$+$

So the solution to $x^3 + 2x^2 - 9x - 18 > 0$ is $-3 < x < -2$ or $x > 3 \Leftrightarrow$
$x \in (-3, -2) \cup (3, \infty)$.

(o) $x^3 - 8 \geq 0 \Rightarrow x^3 \geq 8 \Rightarrow x \geq 2 \Leftrightarrow x \in [2, \infty)$

(p) $x^9 + x > 0 \Rightarrow x(x^8 + 1) > 0 \Rightarrow x > 0$ since $x^8 + 1$ is always positive $\Rightarrow x \in (0, \infty)$

2. (a) $\frac{2x+1}{x^2+1} > 0 \Rightarrow 2x + 1 > 0$ since $x^2 + 1 > 0$ for all x. So $2x > -1 \Rightarrow x > -\frac{1}{2} \Leftrightarrow$
$x \in (-\frac{1}{2}, \infty)$.

(b) $\frac{x+2}{x-3} > 0 \Rightarrow x + 2 > 0$ and $x - 3 > 0$ or $x + 2 < 0$ and $x - 3 < 0 \Rightarrow x > -2$ and $x > 3$ or $x < -2$ and $x < 3 \Rightarrow x > 3$ or $x < -2 \Leftrightarrow x \in (-\infty, -2) \cup (3, \infty)$

(c) $\frac{x^2 + x}{(x-1)^3} < 0 \Rightarrow x^2 + x > 0$ and $x - 1 < 0$ or $x^2 + x < 0$ and $x - 1 > 0 \Rightarrow x(x+1) > 0$ and $x < 1$ or $x(x+1) < 0$ and $x > 1 \Rightarrow$ [$x > 0$ and $x + 1 > 0$ or $x < 0$ and $x + 1 < 0$] and $x < 1$ or [$x > 0$ and $x + 1 < 0$ or $x < 0$ and $x + 1 > 0$] and $x > 1$
$\Rightarrow$ [$x > 0$ and $x > -1$ or $x < 0$ and $x < -1$] and $x < 1$ or
[$x > 0$ and $x < -1$ or $x < 0$ and $x > -1$] and $x > 1$
$\Rightarrow$ [$x > 0$ or $x < -1$] and $x < 1$ or [$-1 < x < 0$] and $x > 1 \Rightarrow x < -1$ or $0 < x < 1 \Leftrightarrow$
$x \in (-\infty, -1) \cup (0, 1)$

(d) $\frac{5x}{(x^2-1)^2} < 0 \Rightarrow 5x < 0 \Rightarrow x < 0$ since $(x^2 - 1)^2 \geq 0$ for all x, so $x \neq \pm 1$.
Thus the solution is $x \in (-\infty, -1) \cup (-1, 0)$.

Exercise 4.1

1. From the graphs

(a) Increases on $(-5,0)$ and $(2,5)$. Decreases on $(0,2)$

(b) Increases on $(-4,3)$. Decreases on $(-6,-4)$ and $(3,5)$

2. (a) $f(x) = 12 + x - x^2$; $f'(x) = 1 - 2x > 0$ when $x < \frac{1}{2}$. So f is increasing on $(-\infty, \frac{1}{2})$

(b) $f(x) = x^4$; $f'(x) = 4x^3 > 0$ when $x > 0$. So f is increasing on $(0, \infty)$.

(c) $g(x) = x^3 - 3x + 2$; $g'(x) = 3x^2 - 3 = 3(x^2 - 1) > 0$ when $x > 1$ or $x < -1$. So g is increasing on $(-\infty, -1)$ and $(1, \infty)$.

(d) $g(x) = 2x^3 - 3x^2$; $g'(x) = 6x^2 - 6x > 0 \Rightarrow x(x-1) > 0 \Rightarrow x > 0$ and $x - 1 > 0$ or $x < 0$ and $x - 1 < 0 \Rightarrow x > 0$ and $x > 1$ or $x < 0$ and $x < 1 \Rightarrow x > 1$ or $x < 0$. So g is increasing on $(-\infty, 0)$ and $(1, \infty)$.

(e) $y = 3x^4 + 4x^3 - 12x^2 + 7$; $y' = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2)$
$= 12x(x+2)(x-1) = 0$ when $x = 0$, $x = -2$, or $x = 1$. This divides the real line into four intervals, $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, and $(1, \infty)$.

Interval	$x+2$	x	$x-1$	y'	y
$x < -2$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -2)$
$-2 < x < 0$	$+$	$-$	$-$	$+$	increasing on $(-2, 0)$
$0 < x < 1$	$+$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$+$	increasing on $(1, \infty)$

(f) $y = x^5 + 8x^3 + x$; $y' = 5x^4 + 24x^2 + 1$. Since $y' > 0$ for all x, y is increasing on $(-\infty, \infty)$.

3. (a) $f(x) = x^2 + x^3$; $f'(x) = 2x + 3x^2 < 0 \Rightarrow x(2 + 3x) < 0 \Rightarrow x < 0$ and $2 + 3x > 0$ or $x > 0$ and $2 + 3x > 0 \Rightarrow x < 0$ and $x > -\frac{2}{3}$ or $x > 0$ and $x < -\frac{2}{3} \Rightarrow -\frac{2}{3} < x < 0$.
So f is decreasing on $(-\frac{2}{3}, 0)$.

(b) $g(x) = 2x^3 - 3x^2 - 36x + 62$; $g'(x) = 6x^2 - 6x - 36 < 0 \Rightarrow (x-3)(x+2) < 0 \Rightarrow$ $x - 3 < 0$ and $x + 2 > 0$ or $x - 3 > 0$ and $x + 2 < 0 \Rightarrow x < 3$ and $x > -2$ or $x > 3$ and $x < -2 \Rightarrow -2 < x < 3$. So g is decreasing on $(-2, 3)$.

(c) $h(x) = (1 - x^2)^2$; $h'(x) = -4x(1 - x^2) < 0 \Rightarrow x(1 - x^2) > 0 \Rightarrow x > 0$ and $1 - x^2 > 0$ or $x < 0$ and $1 - x^2 < 0 \Rightarrow x > 0$ and $x^2 < 1$ or $x < 0$ and $x^2 > 1 \Rightarrow x > 0$ and $-1 < x < 1$ or $x < 0$ and $[x < -1$ or $x > 1] \Rightarrow 0 < x < 1$ or $x < -1$. So h is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(d) $F(x) = 4x + x^4$; $F'(x) = 4 + 4x^3 < 0$ when $x < -1$. So F is decreasing on $(-\infty, -1)$.

4. (a) $f(x) = 3x^2 - 18x + 1$; $f'(x) = 6x - 18$. $6x - 18 > 0$ when $x > 3$, $6x - 18 < 0$ when $x < 3$. So F is increasing on $(3, \infty)$ and is decreasing on $(-\infty, 3)$.

(b) $f(x) = 2x^3 - 9x^2 - 60x + 82$; $f'(x) = 6x^2 - 18x - 60 > 0 \Rightarrow (x-5)(x+2) > 0 \Rightarrow$ $x - 5 > 0$ and $x + 2 > 0$ or $x - 5 < 0$ and $x + 2 < 0 \Rightarrow x > 5$ and $x > -2$ or $x < 5$ and $x < -2 \Rightarrow x > 5$ or $x < -2$. So f is increasing on $(-\infty, -2)$ and $(5, \infty)$ and is decreasing on $(-2, 5)$.

(c) $g(x) = x^4 - 2x^2 + 16$; $g'(x) = 4x^3 - 4x > 0 \Rightarrow x(x^2 - 1) > 0 \Rightarrow x > 0$ and $x^2 - 1 > 0$ or $x < 0$ and $x^2 - 1 < 0 \Rightarrow x > 0$ and $[x > 1$ or $x < -1]$ or $x < 0$ and $-1 < x < 1 \Rightarrow x > 1$ or $-1 < x < 0$. So g is increasing on $(-1, 0)$ and $(1, \infty)$ and is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(d) $g(x) = 3x^4 - 16x^3 + 6x^2 + 72x + 8$; $g'(x) = 12x^3 - 48x^2 + 12x + 72$ $= 12(x^3 - 4x^2 + x + 6)$. If $12(x^3 - 4x^2 + x + 6) = 0$, then $x^3 - 4x^2 + x + 6 = 0$, so $x + 1$ is a factor.

$$
\begin{array}{r l}
 & x^2 - 5x + 6 \\
x + 1 \,\big) & \overline{x^3 - 4x^2 + x + 6} \\
 & \underline{x^3 + x^2} \\
 & \quad -5x^2 + x \\
 & \quad \underline{-5x^2 - 5x} \\
 & \qquad\quad 6x + 6 \\
 & \qquad\quad \underline{6x + 6}
\end{array}
$$

$x^3 - 4x^2 + x + 6 = (x+1)(x^2 - 5x + 6)$
$= (x+1)(x-2)(x-3)$

$(x+1)(x-2)(x-3) = 0$ when $x = -1$, $x = 2$, and $x = 3$.

Interval	$x+1$	$x-2$	$x-3$	g'	g
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 2$	$+$	$-$	$-$	$+$	increasing on $(-1, 2)$
$2 < x < 3$	$+$	$+$	$-$	$-$	decreasing on $(2, 3)$
$x > 3$	$+$	$+$	$+$	$+$	increasing on $(3, \infty)$

(e) $h(x) = x^3(x-1)^4$; $h'(x) = 4x^3(x-1)^3 + 3x^2(x-1)^4 = x^2(x-1)^3[4x + 3(x-1)]$ $= x^2(x-1)^3(7x - 3) > 0$ when $(x-1)(7x-3) > 0 \Rightarrow x - 1 > 0$ and $7x - 3 > 0$ or $x - 1 < 0$ and $7x - 3 < 0 \Rightarrow x > 1$ and $x > \frac{3}{7}$ or $x < 1$ and $x < \frac{3}{7} \Rightarrow x > 1$ or $x < \frac{3}{7}$. So h is increasing on $(-\infty, \frac{3}{7})$ and $(1, \infty)$ and is decreasing on $(\frac{3}{7}, 1)$.

(f) $h(x) = \dfrac{x-1}{x+1}$; $h'(x) = \dfrac{x+1-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$ $(x \neq -1)$. Since $h'(x) > 0$ for all $x \neq -1$, h is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

(g) $y = x\sqrt{4-x}$; $y' = \dfrac{-x}{2\sqrt{4-x}} + \sqrt{4-x} = \dfrac{1}{\sqrt{4-x}}(-\frac{1}{2}x + 4 - x) = \dfrac{4-\frac{3}{2}x}{\sqrt{4-x}}$ $(x<4)$.

So $y'(x)>0$ when $4-\frac{3}{2}x>0 \Rightarrow x<\frac{8}{3}$. So y is increasing on $(-\infty,\frac{8}{3})$ and is decreasing on $(\frac{8}{3},4)$.

(h) $y = (x^2-9)^{\frac{2}{3}}$; $y' = \frac{4}{3}x(x^2-9)^{-\frac{1}{3}}$ $(x \neq \pm 3)$. So $y'>0 \Rightarrow x(x^2-9)^{-\frac{1}{3}}>0 \Rightarrow x>0$ and $x^2-9>0$ or $x<0$ and $x^2-9<0 \Rightarrow x>0$ and $[x>3$ or $x<-3]$ or $x<0$ and $-3<x<3$ $\Rightarrow x>3$ or $-3<x<0$. So y is increasing on $(-3,0)$ and $(3,\infty)$ and is decreasing on $(-\infty,-3)$ and $(0,3)$.

5. $y = 12x^5+15x^4-20x^3+27$; $y' = 60x^4+60x^3-60x^2 = 60x^2(x^2+x-1) = 0$ when $x=0$ or $x^2+x-1=0 \Rightarrow x = \dfrac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)} = \dfrac{-1\pm\sqrt{5}}{2}$.

So $x^2+x-1 = (x-\frac{-1+\sqrt{5}}{2})(x-\frac{-1-\sqrt{5}}{2})<0$ when $x-\frac{-1+\sqrt{5}}{2}>0$ and

$x-\frac{-1-\sqrt{5}}{2}<0$ or $x-\frac{-1+\sqrt{5}}{2}<0$ and $x-\frac{-1-\sqrt{5}}{2}>0 \Rightarrow x>\frac{-1+\sqrt{5}}{2}$ and

$x<\frac{-1-\sqrt{5}}{2}$ or $x<\frac{-1+\sqrt{5}}{2}$ and $x>\frac{-1-\sqrt{5}}{2} \Rightarrow \frac{-1-\sqrt{5}}{2}<x<\frac{-1+\sqrt{5}}{2}$.

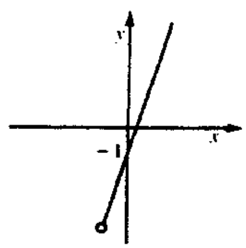
So y is decreasing on $(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$.

Exercise 4.2

1. From the graph:

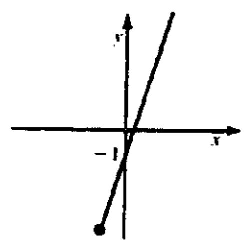
	(i)	(ii)
(a) Absolute maximum value	$f(7) = 5$	$f(4) = 5$
(b) Absolute minimum value	$f(2) = -2$	$f(-3) = -2$
(c) Local maximum values	$f(0) = 2$, $f(4) = 3$	$f(1) = 4$, $f(4) = 5$
(d) Local minimum values	$f(2) = -2$, $f(6) = 1$	$f(2) = 2$, $f(6) = 1$

2. (a) $f(x) = 3x - 1,\ x > -1$



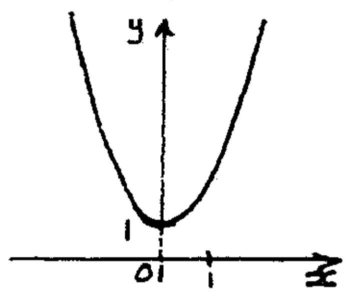
No absolute, or local, maximum or minimum values.

(b) $g(x) = 3x - 1,\ x \geq -1$



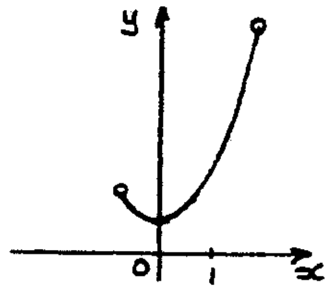
Absolute minimum value $g(-1) = -4$. No maximum values.

(c) $f(x) = x^2 + 1$



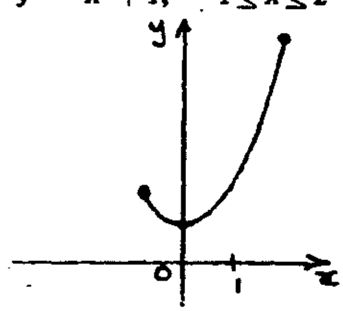
Absolute and local minimum value $f(0) = 1$. No maximum values.

(d) $y = x^2 + 1,\ -1 < x < 2$



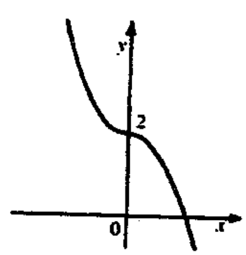
Absolute and local minimum value $f(0) = 1$. No maximum values.

(e) $y = x^2 + 1,\ -1 \leq x \leq 2$



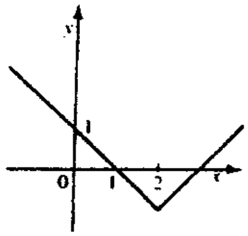
Absolute and local minimum value $f(0) = 1$. Absolute maximum value $f(2) = 5$.

(f) $y = 2 - x^3$



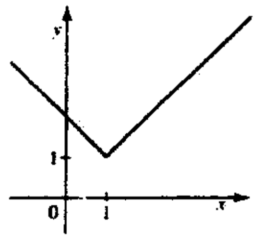
No absolute, or local, maximium or minimum values.

(g) $y = |x-2|-1$



Absolute and local minimum value $f(2) = -1$. No maximum values.

(h) $f(x) = \begin{cases} 2-x & \text{if } x<1 \\ x & \text{if } x>1 \end{cases}$



Absolute and local minimum value $f(1) = 1$. No maximum values.

3. (a) $f(x) = 17-6x+12x^2$; $f'(x) = 24x-6$. f' exists for all x.
$f'(x) = 0 \Rightarrow 24x-6 = 0$ when $x = \frac{1}{4}$. So $x=\frac{1}{4}$ is the only critical number.
(b) $f(x) = x^3-3x+2$; $f'(x) = 3x^2-3$. f' exists for all x. $f'(x) = 0 \Rightarrow 3x^2-3 = 0$ when $x = \pm 1$. So $x=1$ and $x=-1$ are the critical numbers.
(c) $g(x) = x^4-4x^3-8x^2-1$; $g'(x) = 4x^3-12x^2-16x$. g' exists for all x.
$g'(x) = 0 \Rightarrow 4x(x^2-3x-4) = 0 \Rightarrow x(x+1)(x-4) = 0 \Rightarrow x=0, x=-1, x=4$.
So the critical numbers are $x=0$, $x=-1$, $x=4$.
(d) $g(x) = 3x^4-16x^3+6x^2+72x+8$; $g'(x) = 12x^3-48x^2+12x+72$.
g' exists for all x. $g'(x) = 0 \Rightarrow 12(x^3-4x^2+x+6) = 0$; $x+1$ is a factor.

$$\begin{array}{r} x^2-5x+6 \\ x+1\ \overline{)\,x^3-4x^2+x+6} \\ \underline{x^3+x^2} \qquad\qquad \\ -5x^2+x \qquad \\ \underline{-5x^2-5x} \qquad \\ 6x+6 \\ \underline{6x+6} \end{array}$$

$x^3-4x^2+x+6 = (x+1)(x^2-5x+6)$
$= (x+1)(x-2)(x-3)$

So $g'(x) = 0$ when $x=-1$, $x=2$, $x=3$.
So the critical numbers are $x=-1$, $x=2$, and $x=3$.

(e) $y = 2x^3+3x^2-6x+3$; $y' = 6x^2+6x-6$. y' exists for all x. $y' = 0 \Rightarrow$
$6x^2+6x-6 = 0 \Rightarrow x^2+x-1 = 0 \Rightarrow x = \dfrac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)} = \dfrac{-1\pm\sqrt{5}}{2}$.
So the critical numbers are $x = \dfrac{-1+\sqrt{5}}{2}$ and $x = \dfrac{-1-\sqrt{5}}{2}$.
(f) $y = x^3+x^2+x+1$; $y' = 3x^2+2x+1$. The discriminant of this quadratic is $b^2-4ac = 2^2-4(3)(1) = -8 < 0$. Thus $y' = 0$ has no roots, so there are no critical numbers.

(g) $y = |x+6|$. We know from Example 7 in Section 2.1 that $|x|$ is not differentiable at 0, so $|x+6|$ is not differentiable when $x+6 = 0 \Rightarrow x = -6$ by the same reasoning. So $f'(-6)$ doesn't exist, therefore -6 is a critical number.

Since $y = \begin{cases} x+6 & \text{if } x \geq -6 \\ -x-6 & \text{if } x < -6 \end{cases}$; $y' = \begin{cases} 1 & \text{if } x > -6 \\ -1 & \text{if } x < -6 \end{cases}$

So y' is never 0. So the only critical number is -6.

(h) $y = \sqrt[3]{x}$; $y' = \dfrac{1}{3\sqrt[3]{x^2}}$. y' doesn't exist when $x = 0$, y' never equals 0.

So 0 is the only critical number.

(i) $y = x - \sqrt{x}$; $y' = 1 - \dfrac{1}{2\sqrt{x}}$. y' exists for all $x > 0$. $y' = 0 \Rightarrow 1 - \dfrac{1}{2\sqrt{x}} = 0 \Rightarrow$ $\sqrt{x} = \frac{1}{2} \Rightarrow x = \frac{1}{4}$. So 0 and $\frac{1}{4}$ are the only critical numbers.

(j) $y = x\sqrt{x-1}$; $y' = \dfrac{x}{2\sqrt{x-1}} + \sqrt{x-1} = \dfrac{1}{\sqrt{x-1}}(\frac{x}{2} + x - 1) = \dfrac{1}{\sqrt{x-1}}(\frac{3x}{2} - 1)$.

y' exists for all $x > 1$. $y' = 0 \Rightarrow \dfrac{1}{\sqrt{x-1}}(\frac{3x}{2} - 1) = 0 \Rightarrow \frac{3x}{2} - 1 = 0 \Rightarrow x = \frac{2}{3}$.

So $\frac{2}{3}$ and 1 are the only critical numbers.

(k) $y = \dfrac{t}{t+1}$; $y' = \dfrac{t+1-t}{(t+1)^2} = \dfrac{1}{(t+1)^2}$. y' exists for all $t \neq -1$. $y' = 0 \Rightarrow$ $\dfrac{1}{(t+1)^2} = 0$ to which there are no solutions. Since -1 is not in the domain of y, it cannot be a critical number, so there are no critical numbers.

(l) $y = \dfrac{t}{t^2+1}$; $y' = \dfrac{t^2+1-2t^2}{(t^2+1)^2} = \dfrac{1-t^2}{(t^2+1)^2}$. y' exists for all t.

$y' = 0 \Rightarrow 1 - t^2 = 0 \Rightarrow t = \pm 1$. So 1 and -1 are critical numbers.

4. (a) $f(x) = 2x^2 - 8x + 1$, $0 \leq x \leq 3$

$f'(x) = 4x - 8$; f' exists for all x. $f'(x) = 0$ when $x = 2$. So 2 is the only critical number. $f(2) = -7$, $f(0) = 1$, $f(3) = -5$. So the absolute maximum value is $f(0) = 1$ and the absolute minimum value is $f(2) = -7$.

(b) $f(x) = 3 + 2(x+1)^2$, $-3 \leq x \leq 2$

$f'(x) = 4x + 4$; f' exists for all x. $f'(x) = 0$ when $x = -1$. So -1 is the only critical number. $f(-1) = 3$, $f(-3) = 11$, $f(2) = 21$. So the absolute maximum value is $f(2) = 21$ and the absolute minimum value is $f(-1) = 3$.

(c) $f(x) = 2x^3 - 3x^2$, $-2 \leq x \leq 2$

$f'(x) = 6x^2 - 6x$; f' exists for all x. $f'(x) = 0$ when $x = 0$, $x = 1$. So 0 and 1 are the critical numbers. $f(0) = 0$, $f(1) = -1$, $f(-2) = -28$, $f(2) = 4$. So the absolute maximim value is $f(2) = 4$ and the absolute minimum value is $f(-2) = -28$.

(d) $f(x) = 2x^3 - 3x^2 - 36x + 62, \ -3 \leq x \leq 4$

$f'(x) = 6x^2 - 6x - 36$; f' exists for all x. $f'(x) = 0$ when $x^2 - x - 6 = 0 \Rightarrow$ $(x-3)(x+2) = 0 \Rightarrow x = 3, x = -2$. So the critical numbers are 3 and -2. $f(3) = -19$, $f(-2) = 106$, $f(-3) = 89$, $f(4) = -2$. So the absolute maximum value is $f(-2) = 106$ and the absolute minimum value is $f(3) = -19$.

(e) $f(x) = x^4 - 2x^2 + 16, \ -3 \leq x \leq 2$

$f'(x) = 4x^3 - 4x$; f' exists for all x. $f'(x) = 0$ when $x = -1$, $x = 0$, $x = 1$. So the critical numbers are $-1, 0, 1$. $f(-1) = 15$, $f(0) = 16$, $f(1) = 15$, $f(-3) = 79$, $f(2) = 24$. So the absolute maximum value is $f(-3) = 79$ and the absolute minimum value is $f(1) = f(-1) = 15$.

(f) $f(x) = x^5 + 3x^3 + x, \ -1 \leq x \leq 2$

$f'(x) = 5x^4 + 9x^2 + 1$; f' exists for all x. $f'(x) > 0$ for all x, so there are no critical numbers. $f(-1) = -5$, $f(2) = 58$. So the absolute maximum value is $f(2) = 58$ and the absolute minimum value $f(-1) = -5$.

(g) $g(x) = x^2 + \frac{16}{x}, \ 1 \leq x \leq 4$

$g'(x) = 2x - \frac{16}{x^2}$; g' exists for all $x \neq 0$. $g'(x) = 0 \Rightarrow 2x - \frac{16}{x^2} = 0 \Rightarrow 2x^3 = 16 \Rightarrow$ $x = 2$. Since 0 is not in the domain of g, it is not a critical number. So the only critical number is 2. $g(2) = 12$, $g(1) = 17$, $g(4) = 20$. So the absolute maximum value is $g(4) = 20$ and the absolute minimum value is $g(2) = 12$.

(h) $f(x) = 3\sqrt[3]{x^2} - 2x, \ 1 \leq x \leq 3$

$f'(x) = \frac{2}{\sqrt[3]{x}} - 2$; f' exists for all $x \neq 0$. $f'(x) = 0 \Rightarrow 2\sqrt[3]{x} = 2 \Rightarrow x = 1$.

So 1 and 0 are critical numbers, but 0 is not in the given interval.

$f(1) = 1$, $f(3) = 3\sqrt[3]{9} - 6$. So the absolute maximum value is $f(1) = 1$ and the absolute minimum value is $f(3) = 3\sqrt[3]{9} - 6 \doteq 0.24$.

(i) $f(x) = \sqrt[3]{(x^2-9)^2}, \ -6 \leq x \leq 6$

$f'(x) = \frac{4x}{3\sqrt[3]{(x^2-9)^2}}$; f' exists for all x except $x = 3$, $x = -3$. $f'(x) = 0$ when $x = 0$. So -3, 0, and 3 are the critical numbers. $f(\pm 3) = 0$, $f(0) = \sqrt[3]{81}$, $f(\pm 6) = 9$. So the absolute maximum value is $f(\pm 6) = 9$, and the absolute minimum value is $f(\pm 3) = 0$.

(j) $f(x) = |2x - 1| - 1, \ 0 \leq x \leq 2$

We know from Example 7 in Section 2.1 that $|x|$ is not differentiable at 0, so $f'(x)$ will not exist where $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$. So $\frac{1}{2}$ is a critical number.

Exercise 4.2

Since $f(x) = \begin{cases} 2x-1-1 & \text{if } x \geq \frac{1}{2} \\ -2x+1-1 & \text{if } x < \frac{1}{2} \end{cases}$; $f'(x) = \begin{cases} 2 & \text{if } x > \frac{1}{2} \\ -2 & \text{if } x < \frac{1}{2} \end{cases}$

So $f'(x)$ is never 0. So $\frac{1}{2}$ is the only critical number. $f(0) = 0$, $f(\frac{1}{2}) = -1$, $f(2) = 2$. So the absolute maximum value is $f(2) = 2$, and the absolute minimum value is $f(\frac{1}{2}) = -1$.

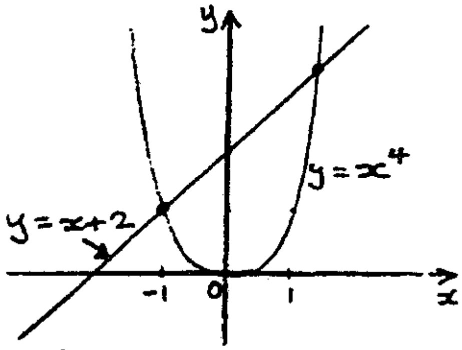
5. Show $y = x^{21} + x^{11} + 13x$ has no local maximum or minimum values. So we must show that there are no critical numbers. $y' = 21x^{20} + 11x^{10} + 13$; y' exists for all x and $y' > 0$ for all x. So there are no critical numbers; therefore, there are no local maximum or minimum values.

6. $y = x^2 + kx + 72$, find k if there is a local minimum at $x = 4$.
$y' = 2x + k$; y' exists for all x. So $2x + k = 0$ when $x = 4 \Rightarrow k = -8$.

7. $y = 2x^3 + ax^2 + bx + 36$, find a and b if there is a local maximum when $x = -4$ and a local minimum when $x = 5$. $y' = 6x^2 + 2ax + b$; y' exists for all x. So $6x^2 + 2ax + b = 0$ when $x = -4$ and when $x = 5$. So

$$6(-4)^2 + 2a(-4) + b = 0 \Rightarrow b - 8a = -96$$
$$6(5)^2 + 2a(5) + b = 0 \Rightarrow \underline{b + 10a = -150} \text{ (subtract)}$$
$$-18a = 54$$
$$a = -3 \Rightarrow b - 8(-3) = -96 \Rightarrow b = -120$$

8. (a) Find critical numbers of $f(x) = 2x^5 - 5x^2 - 20x + 12$ to 3 decimal places. $f'(x) = 10x^4 - 10x - 20$; f' exists for all x. $f'(x) = 0 \Rightarrow x^4 = x + 2$. So to get the approximations to the values of the critical numbers, we must draw $y = x^4$ and $y = x + 2$ on the same graph, and use the intersection points as our approximations.



From the graph, we see that $f'(x) = 0$ has two roots. We will use $x = 1$ and $x = -1$ as our approximations. We must remember that we are trying to find the solution to $f'(x) = 0$, so we must take $f''(x)$ as well to use Newton's method. $f''(x) = 40x^3 - 10$,

so $x_{n+1} = x_n - \dfrac{10x_n^4 - 10x_n - 20}{40x_n^3 - 10}$

Exercise 4.2

Guess $x_1 = -1$

$x_2 = -1.000$

Guess $x_1 = 1$

$x_2 \doteq 1.667$

$x_3 \doteq 1.436$

$x_4 \doteq 1.361$

$x_5 \doteq 1.353$

$x_6 \doteq 1.353$

So the critical numbers are -1 and 1.353 to three decimal places.

(b) $f(x) = 2x^5 - 5x^2 - 20x + 12, \ -1 \leq x \leq 2$.

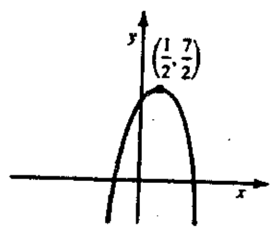
From (a), the critical numbers are -1, 1.353.

$f(-1) = 25$, $f(1.353) = -15.14$, $f(2) = 16$. So the absolute minimum value is $f(1.353) = -15.14$ to two decimal places.

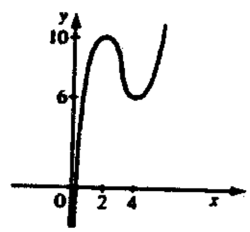
Exercise 4.3

1. (a) $f(x) = 3x^2 - 4x + 13$; $f'(x) = 6x - 4$, $f'(x) = 0$ when $x = \frac{2}{3}$.
When $x < \frac{2}{3}$, $f'(x) < 0$, and when $x > \frac{2}{3}$, $f'(x) > 0$. So $f(\frac{2}{3}) = \frac{35}{3}$ is a local minimum.
(b) $f(x) = x^3 - 12x - 5$; $f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$, $f'(x) = 0$ when $x = \pm 2$. When $|x| < 2$, $f'(x) < 0$, and when $|x| > 2$, $f'(x) > 0$.
So $f(-2) = 11$ is a local maximum and $f(2) = -21$ is a local minimum.
(c) $f(x) = 2 + 5x - x^5$; $f'(x) = 5 - 5x^4 = 5(1 - x^4)$, $f'(x) = 0$ when $x = \pm 1$.
When $|x| < 1$, $f'(x) > 0$, and when $|x| > 1$, $f'(x) < 0$. So $f(1) = 6$ is a local maximum and $f(-1) = -2$ is a local minimum.
(d) $f(x) = x^4 - x^3$; $f'(x) = 4x^3 - 3x^2 = x^2(4x - 3)$, $f'(x) = 0$ when $x = 0$ and $x = \frac{3}{4}$. When $x < 0$, $f'(x) < 0$, when $0 < x < \frac{3}{4}$, $f'(x) < 0$, and when $x > \frac{3}{4}$, $f'(x) > 0$.
So $f(\frac{3}{4}) = -\frac{27}{256}$ is a local minimum.

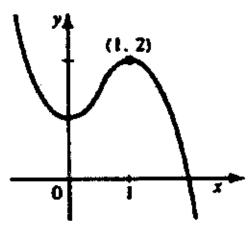
2. (a) $f(x) = 2 + 6x - 6x^2$;
$f'(x) = -12x + 6 = 0$ when $x = \frac{1}{2}$. So critical number is $\frac{1}{2}$. $f'(x) > 0$ when $x < \frac{1}{2}$ and $f'(x) < 0$ when $x > \frac{1}{2}$, so the function increases on $(-\infty, \frac{1}{2})$, and decreases on $(\frac{1}{2}, \infty)$.
Thus, the local maximum is $f(\frac{1}{2}) = \frac{7}{2}$.



(b) $f(x) = x^3 - 9x^2 + 24x - 10$;
$f'(x) = 3x^2 - 18x + 24 = 3(x-4)(x-2) = 0$ when $x = 2, 4$. So critical numbers are 2 and 4. $f'(x) > 0$ when $x < 2$ and when $x > 4$.
$f'(x) < 0$ when $2 < x < 4$, so the function increases on $(-\infty, 2)$ and $(4, \infty)$, and decreases on $(2,4)$. Thus, the local maximum is $f(2) = 10$, local minimum is $f(4) = 6$.



(c) $g(x) = 1 + 3x^2 - 2x^3$;
$g'(x) = 6x - 6x^2 = 6x(1 - x) = 0$ when $x = 0, 1$. So critical numbers are 0 and 1.
$f'(x) > 0$ when $0 < x < 1$ and $f'(x) < 0$ when $x < 0$ and when $x > 1$, so the function increases on $(0,1)$, and decreases on $(-\infty, 0)$ and $(1, \infty)$. The local maximum is $g(1) = 2$ and local minimum is $g(0) = 1$.

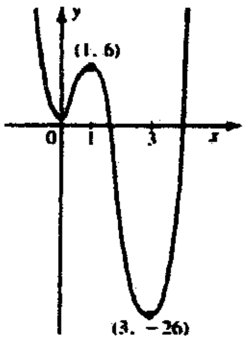


(d) $g(x) = 3x^4 - 16x^3 + 18x^2 + 1$;

$g'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x-1)(x-3) = 0$ when $x = 0$, 1, or 3. So critical numbers are 0, 1, and 3.

Interval	x	x − 1	x − 3	g′	g
$(-\infty,0)$	−	−	−	−	decreasing
(0,1)	+	−	−	+	increasing
(1,3)	+	+	−	−	decreasing
$(3,\infty)$	+	+	+	+	increasing

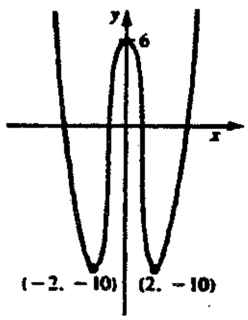
The a local maximum is $g(1) = 6$ and local minima are $g(0) = 1$ and $g(3) = -26$.



(e) $h(x) = x^4 - 8x^2 + 6$; $h'(x) = 4x^3 - 16x = 4x(x-2)(x+2) = 0$ when $x = -2$, 0, or 2. So critical numbers are -2, 0, and 2.

Interval	x	x − 2	x + 2	h′	h
$(-\infty,-2)$	−	−	−	−	decreasing
$(-2,0)$	−	−	+	+	increasing
(0,2)	+	−	+	−	decreasing
$(2,\infty)$	+	+	+	+	increasing

The local maximum is $h(0) = 6$ and local minima are $h(\pm 2) = -10$.

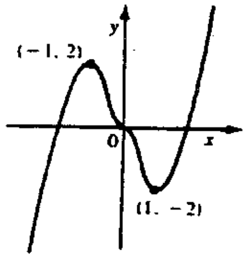


(f) $h(x) = 3x^6 - 5x^3$; $h'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1) = 0$ when $x = -1$, 0, or 1. So critical numbers are -1, 0 and 1.

Exercise 4.3

Interval	x^2	$x-1$	$x+1$	h'	h
$(-\infty,-1)$	+	−	−	+	increasing
$(-1,0)$	+	−	+	−	decreasing
$(0,1)$	+	−	+	−	decreasing
$(1,\infty)$	+	+	+	+	increasing

The local maximum is $h(-1) = 2$ and local minimum is $h(1) = -2$.



3. (a) $f(x) = 2x^{\frac{2}{3}}(3-4x^{\frac{1}{3}}) = 6x^{\frac{2}{3}} - 8x$; $f'(x) = 4x^{-\frac{1}{3}} - 8 = 0$ when $x^{-\frac{1}{3}} = 2$, so $x = \frac{1}{8}$.

So the critical numbers are 0 and $\frac{1}{8}$. For $x < 0$ and for $x > \frac{1}{8}$, $f'(x) < 0$. For $0 < x < \frac{1}{8}$, $f'(x) > 0$. So $f(\frac{1}{8}) = \frac{1}{2}$ is a local maximum and $f(0) = 0$ is a local minimum.

(b) $f(x) = \dfrac{x^2}{x^2-1}$; $f'(x) = \dfrac{2x(x^2-1)-2x^3}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$. $(x^2-1)^2 \geq 0$, so f and f′ are not defined for $x = \pm 1$ and $f'(x) > 0$ on $(-\infty,-1)$ and $(-1,0)$ and $f'(x) < 0$ on $(0,1)$ and $(1,\infty)$, so $f(0) = 0$ is a local maximum.

(c) $f(x) = x\sqrt{4-x}$; $f'(x) = \sqrt{4-x} - \dfrac{x}{2\sqrt{4-x}} = \dfrac{2(4-x)-x}{2\sqrt{4-x}} = \dfrac{8-3x}{2\sqrt{4-x}}$.

$f'(x) > 0$ for $x < \frac{8}{3}$ and $f'(x) < 0$ for $x > \frac{8}{3}$, so $f(\frac{8}{3}) = \dfrac{16\sqrt{3}}{9}$ is a local maximum.

(d) $f(x) = x\sqrt{1-x^2}$; $f'(x) = \sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}} = \dfrac{1-2x^2}{\sqrt{1-x^2}}$. $f'(x) > 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f'(x) < 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$, so $f(\frac{1}{\sqrt{2}}) = \frac{1}{2}$ is a local maximum and $f(-\frac{1}{\sqrt{2}}) = -\frac{1}{2}$ is a local minimum.

4. (a) $f(x) = 27 + x - x^2$; $f'(x) = 1 - 2x$.
$f'(x) < 0$ for $x > \frac{1}{2}$ and $f'(x) > 0$ for $x < \frac{1}{2}$, so $f(\frac{1}{2}) = \frac{109}{4}$ is an absolute maximum.

(b) $f(x) = 3 - \dfrac{1}{\sqrt{x^2+1}}$; $f'(x) = \dfrac{x}{(x^2+1)^{\frac{3}{2}}}$.
$f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, so $f(0) = 2$ is an absolute minimum.

Exercise 4.3

(c) $g(x) = \dfrac{x^2-1}{x^2+1}$; $g'(x) = \dfrac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

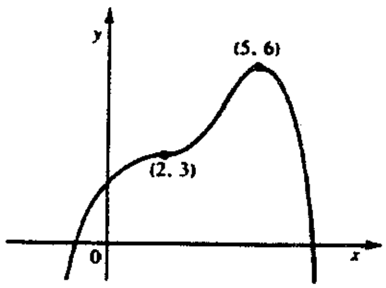
$g'(x)<0$ for $x<0$ and $g'(x)>0$ for $x>0$, so $g(0) = -1$ is an absolute minimum.

(d) $g(x) = \dfrac{x^2-x+1}{x^2+1}$, $x \geq 0$; $g'(x) = \dfrac{(2x-1)(x^2+1)-2x(x^2-x+1)}{(x^2+1)^2}$

$= \dfrac{2x^3-x^2+2x-1-2x^3+2x^2-2x}{(x^2+1)^2} = \dfrac{x^2-1}{(x^2+1)^2}$.

$g'(x)<0$ for $0<x<1$ and $g'(x)>0$ for $x>1$, so $g(1) = \frac{1}{2}$ is an absolute minimum.

5.



6. $f(x) = \begin{cases} -x & \text{if } x<0 \\ 2x^3-15x^2+36x & \text{if } 0 \leq x \leq 4 \\ 216-x & \text{if } x>4 \end{cases}$; $f'(x) = \begin{cases} -1 & \text{if } x<0 \\ 6x^2-30x+36 & \text{if } 0 \leq x \leq 4 \\ -1 & \text{if } x>4 \end{cases}$.

$f'(x)>0$ on $(0,2)$ and $(3,4)$ and $f'(x)<0$ on $(-\infty,0)$, $(2,3)$, and $(4,\infty)$.

So $f(2) = 28$ is a local maximum and $f(0) = 0$ and $f(3) = 27$ are local minima.

Exercise 4.4

1. Let x and y be the numbers. $P = xy$, $x - y = 150$, so $y = x - 150$. Thus $P = x(x-150) = x^2 - 150x$; $P'(x) = 2x - 150 = 0$ when $x = 75$ and $y = -75$. Since $P'(x) < 0$ when $x < 75$ and $P'(x) > 0$ when $x > 75$, P has an absolute minimum at $x = 75$. So the numbers are 75 and -75.

2. Let x and y be the numbers. $S = x + 2y$, $xy = 200$, so $y = \frac{200}{x}$. Thus $S(x) = x + \frac{400}{x}$; $S'(x) = 1 - \frac{400}{x^2} = \frac{x^2 - 400}{x^2} = 0$ when $x = \pm 20$, but, the numbers must be positive so $x = 20$ and $y = 10$. Since $S'(x) < 0$ when $0 < x < 20$ and $S'(x) > 0$ when $x > 20$, S has an absolute minimum at $x = 20$. So the numbers are 20 and 10.

3. Let L be the length and W be the width. $A = LW$, $2L + 2W = 100$, so $L = 50 - W$. Thus $A(W) = (50 - W)W = 50W - W^2$; $A'(W) = 50 - 2W = 0$ when $W = 25$ and $L = 25$. Since $A''(W) > 0$ when $W < 25$ and $A'(W) < 0$ when $W > 25$, A has an absolute maximum at $W = 25$. So the length and the width are both 25 cm.

4. Let L be the length and W be the width. $P = 2L + 2W$, $A = LW$, so $L = \frac{A}{W}$. Thus $P(W) = 2W + \frac{2A}{W}$; $P'(W) = 2 - \frac{2A}{W^2} = \frac{2(W^2 - A)}{W^2} = 0$ when $W = \sqrt{A}$. Since $P'(W) < 0$ when $0 < W < \sqrt{A}$ and $P'(W) > 0$ when $W > \sqrt{A}$, P has a minimum at $W = \sqrt{A}$. So $L = \frac{A}{W} = \frac{A}{\sqrt{A}} = \sqrt{A} = W$. Since the length equals the width the rectangle is a square.

5. Let x be the length and the width and let h be the height. Surface Area $= A = x^2 + 4xh$. Volume $= 4000 = x^2h$, so $h = \frac{4000}{x^2}$. Thus $A(x) = x^2 + \frac{16000}{x}$; $A'(x) = 2x - \frac{16000}{x^2} = \frac{2(x^3 - 8000)}{x^2} = 0$ when $x = 20$ and $h = \frac{4000}{x^2} = 10$. Since $A'(x) < 0$ when $x < 20$ and $A'(x) > 0$ when $x > 20$, A has an absolute minimum at $x = 20$. Thus the box should be 20 cm by 20 cm by 10 cm.

6. Let x be the length of the squares cut out, $0 < x < \frac{3}{2}$.
Volume $= V(x) = x(3-2x)^2$; $V'(x) = (3-2x)^2 - 4x(3-2x) = 12x^2 - 24x + 9$ $= 3(4x^2 - 8x + 3) = 3(2x-1)(2x-3) = 0$ when $x = \frac{1}{2}$, $x = \frac{3}{2}$, but $x = \frac{3}{2}$ cannot

yield a valid answer. Since $V'(x)>0$ when $0<x<\frac{1}{2}$ and $V'(x)>0$ when $\frac{1}{2}<x<\frac{3}{2}$, V has an absolute maximum at $\frac{1}{2}$. So $V(\frac{1}{2}) = 2\ m^3$ is the absolute maximum volume.

7. (a) Let x be the width and let y be the length, but there is no need for rope along the shore. $A = xy$, $400 = 2x + y$, so $y = 400 - 2x$. Thus $A(x) = x(400-2x) = 400x - 2x^2$; $A'(x) = 400 - 4x = 0$ when $x = 100$ and $y = 200$. Since $A'(x)>0$ when $x<100$ and $A'(x)<0$ when $x>100$, A has an absolute maximum at $x = 100$. So the rectangle should be 100 m by 200 m.

(b) Now. $A(x) = 400x - 2x^2$ where $0<x\le 50$. there are no critical points in this interval, thus $A(x)$ has its largest value at $x = 50$. The dimensions should be 50 m by 300 m.

8. Let x be the length of the rectangle and of the divider and let y be the width. Length of Fence $= F = 3x + 2y$, $xy = 750000$, so $y = \frac{750000}{x}$. Thus $F(x) = 3x + \frac{1500000}{x}$; $F'(x) = 3 - \frac{1500000}{x^2} = \frac{3x^2 - 1500000}{x^2} = 0$ when $x = \sqrt{500000} = 500\sqrt{2}$ and $y = \frac{750000}{500\sqrt{2}} = 750\sqrt{2}$. Since $F'(x)<0$ when $0<x<500\sqrt{2}$ and $F'(x)>0$ when $x>500\sqrt{2}$, F has a minimum at $x = 500\sqrt{2}$. So the field has dimensions $500\sqrt{2}$ m by $750\sqrt{2}$ with a divider of $500\sqrt{2}$ m.

9. $y = f(x)$ and $y = f(c)$ is a minimum, so $f'(c) = 0$ and for $x<c$, $f'(x)<0$ and for $x>c$, $f'(x)>0$. If $f_2(x) = y = \sqrt{f(x)}$, then $f_2'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$, so $f_2'(c) = \frac{f'(c)}{2\sqrt{f(c)}} = \frac{0}{2\sqrt{f(c)}} = 0$. Since $\sqrt{f(x)}>0$ then $f_2'(x)<0$ for $x<c$ and $f_2'(x)>0$ for $x>c$. Thus $y = \sqrt{f(x)}$ also has a minimum at c.

10. $y = 5x + 4$, letting d be the distance to the origin, $d^2(x) = x^2 + (5x+4)^2 = 26x^2 + 40x + 16$, $\frac{d}{dx}[d^2(x)] = 52x + 40 = 0$ when $x = -\frac{40}{52} = -\frac{10}{13}$. Since $\frac{d}{dx}[d^2(x)]<0$ when $x<-\frac{10}{13}$ and $\frac{d}{dx}[d^2(x)]>0$ when $x>-\frac{10}{13}$, $d^2(x)$ has an absolute minimum at $x = -\frac{10}{13}$. Thus the point closest to the origin is $(-\frac{10}{13},\frac{2}{13})$.

11. $2y = x^2$, letting d be the distance to $(-4,1)$, $d^2(x) = (x+4)^2 + (\frac{x^2}{2}-1)^2 = x^2 + 8x + 16 + \frac{x^4}{4} - x^2 + 1 = \frac{x^4}{4} + 8x + 17$, $\frac{d}{dx}[d^2(x)] = x^3 + 8 = 0$ when $x = -2$. Since $\frac{d}{dx}[d^2(x)]<0$ when $x<-2$ and

$\frac{d}{dx}\left[d^2(x)\right] > 0$ when $x > -2$, $d^2(x)$ has an absolute minimum at $x = -2$. Thus the point closest to $(-4,1)$ is $(-2,2)$.

12. Surface Area $= A = 2\pi r^2 + 2\pi rh$, $V = 1000 = \pi r^2 h$, so $h = \frac{1000}{\pi r^2}$. Thus $A(r) = 2\pi r^2 + \frac{2000}{r}$; $A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2} = 0$ when $\pi r^3 = 500$, so $r = \sqrt[3]{\frac{500}{\pi}} \doteq 5.4$. Since $A'(r) < 0$ when $r < \sqrt[3]{\frac{500}{\pi}}$ and $A'(r) > 0$ when $r > \sqrt[3]{\frac{500}{\pi}}$, A has an absolute minimum at $r = \sqrt[3]{\frac{500}{\pi}}$.

13. (a) Let x be the amount used for the square. Area of square $= A_S = \frac{x^2}{16}$. Area of circle $= A_C = \pi r^2$, but $40 - x = 2\pi r$, so $r = \frac{40-x}{2\pi}$, thus

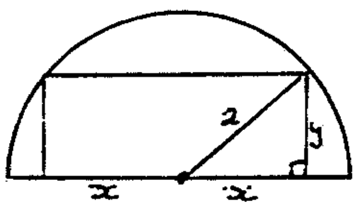
$A_C = \pi\left(\frac{40-x}{2\pi}\right)^2 = \frac{x^2 - 80x + 1600}{4\pi}$.

Total area $= A(x) = \frac{x^2}{16} + \frac{x^2 - 80x + 1600}{4\pi} = \frac{\pi x^2 + 4x^2 - 320x + 6400}{16\pi}$;

$A'(x) = \frac{x}{8} + \frac{x}{2\pi} - \frac{20}{\pi} = 0$ when $x = \frac{160}{4+\pi} \doteq 24.4$. Comparing the values at the endpoints and the critical value, $A(40) = \frac{40^2}{16} = 100\text{ cm}^2$, $A(0) = \frac{40^2}{4\pi} \doteq 127\text{ cm}^2$, and $A(24.4) = \frac{1}{16}(15.6)^2 + \frac{(24.4)^2}{4\pi} \doteq 56.5\text{ cm}^2$. Thus the area is maximized if all the wire is used for the circle.

(b) The area is minimized at $x = \frac{160}{4+\pi} \doteq 24.4$, so there is 24.4 cm for the square and 17.6 cm for the circle.

14. Let 2x be the length of one side, $0 < x < 2$, and let y be the length of the other. Area $= 2xy$, $x^2 + y^2 = 4$, so $y = \sqrt{4-x^2}$, thus, $A(x) = 2x\sqrt{4-x^2}$;



$A'(x) = 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} = \frac{2(4-x^2) - 2x^2}{\sqrt{4-x^2}}$

$= \frac{4(2-x^2)}{\sqrt{4-x^2}} = 0$ when $x = \sqrt{2}$. Since $A'(x) > 0$ when $0 < x < \sqrt{2}$ and $A'(x) < 0$ when $\sqrt{2} < x < 2$, A has a maximum at $x = \sqrt{2}$. Thus the largest area would be $2\sqrt{2}\sqrt{4-2} = 4\text{ cm}^2$.

15. Let C be the total cost. The cost of the underground portion is 40x, $0 \le x \le 1200$. The cost of the underwater portion is $120\sqrt{100^2+(1200-x)^2}$. So

$C(x) = 40x + 120\sqrt{100^2+(1200-x)^2}$; $C'(x) = 40 - \dfrac{120(1200-x)}{\sqrt{100^2+(1200-x^2)}} = 0$ when

$120(1200-x) = 40\sqrt{100^2+(1200-x)^2}$, so $9(1200-x)^2 = 100^2+(1200-x)^2$, then

$8(1200-x)^2 = 100^2$, and $x = 1200 \pm 25\sqrt{2} \doteq 1235$ or 1165. 1235 is not in the domain. Comparing the values at the endpoints and the critical point,

$C(0) = \$144\,499$, $C(1200) = \$60\,000$, and $C(1165) = \$59\,314$. Thus the cheapest method is to lay the cable underground to a point 1165 m east of P and lay the remaining cable underwater.

16. Let h be the height of the rectangle and let x be half the width.

The perimeter $= 8 = 2x + 2h + \pi x$, so $h = \dfrac{8-2x-\pi x}{2} = 4 - x - \frac{1}{2}\pi x$, for

$0 < x \le \dfrac{8}{\pi+2}$. Thus Area $= 2xh + \frac{1}{2}\pi x^2 = 2x(4 - x - \frac{1}{2}\pi x) + \frac{1}{2}\pi x^2$

$= 8x - 2x^2 - \frac{1}{2}\pi x^2$; $A'(x) = 8 - 4x - \pi x = 0$ when $x = \dfrac{8}{\pi+4}$.

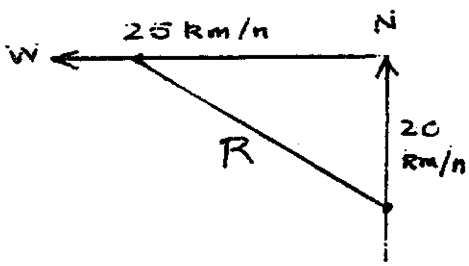
Now, comparing the values at the endpoint and the critical point,

$A\left(\dfrac{8}{\pi+4}\right) = \dfrac{64}{\pi+4} - \dfrac{128}{(\pi+4)^2} - \dfrac{32\pi}{(\pi+4)^2} = \dfrac{32\pi+128}{(\pi+4)^2} = \dfrac{32}{\pi+4} \doteq 4.48$ and

$A\left(\dfrac{8}{\pi+2}\right) = \dfrac{64}{\pi+2} - \dfrac{128}{(\pi+2)^2} - \dfrac{32\pi}{(\pi+2)^2} = \dfrac{32\pi}{(\pi+2)^2} \doteq 3.80$. Thus the largest area

occurs when $x = \dfrac{8}{\pi+4}$, so the base is $\dfrac{16}{\pi+4} \doteq 2.24$ m.

17. Let R be the distance between the boats and let t be the time with $t = 0$ being noon. 25t is the distance travelled by the boat going west and $20 - 20t$ is the distance travelled the boat going north. So,



$R^2 = (25t)^2 + (20-20t)^2$

$= 625t^2 + 400t^2 - 800t + 400$

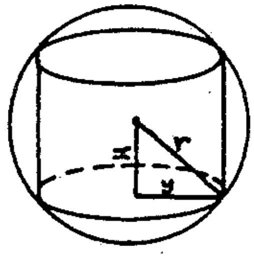
$= 1025t^2 - 800t + 400 = f(t)$;

$f'(t) = 2050t - 800 = 0$ when $t = \dfrac{800}{2050} = \dfrac{16}{41}$.

Since $f'(t) < 0$ when $t < \frac{16}{41}$ and $f'(t) > 0$ when $t > \frac{16}{41}$, f has an absolute minimum at $t = \frac{16}{41}$.

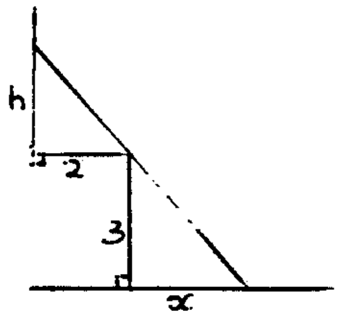
The minimum distance between the boats occurs $\frac{16}{41}$ hours after noon (about 12:23 p.m.).

18. The cylinder has volume $V = \pi y^2(2x)$.
Also $x^2 + y^2 = r^2$, so $y^2 = r^2 - x^2$, thus
$V(x) = \pi(r^2 - x^2)(2x) = 2\pi(r^2x - x^3)$, where
$0 \leq x \leq r$.
$V'(x) = 2\pi(r^2 - 3x^2) = 0$ when $x = \frac{r}{\sqrt{3}}$.
Checking the endpoints $V(0) = V(r) = 0$.
So $V(\frac{r}{\sqrt{3}}) = \frac{4\sqrt{3}\pi r^2}{9}$ is the maximum.



19. Let x be half the width of the track, $0 < x \leq \frac{500}{\pi}$ m, and let y be the length of the straight segment. Area $= 2xy + \pi x^2$. Perimeter $= 1000 = 2\pi x + 2y$, so $y = 500 - \pi x$. Thus $A(x) = 2x(500 - \pi x) + \pi x^2 = 1000x - \pi x^2$;
$A'(x) = 1000 - 2\pi x = 0$ when $x = \frac{500}{\pi}$. This endpoint is the only critical point in the domain. Thus the maximum area occurs when the track is a circle and the radius is $\frac{500}{\pi}$ m or $\frac{1}{2\pi}$ km.

20. Let x be the length along the ground from the corridor to the ladder and let h be the height from the corridor to the ladder. Let L be the square of the length of the ladder. $\frac{2}{h} = \frac{x}{3}$, so $h = \frac{6}{x}$, thus
$L(x) = (x + 2)^2 + (h + 3)^2 = (x + 2)^2 + (\frac{6}{x} + 3)^2$
$= x^2 + 4x + 4 + \frac{36}{x^2} + \frac{36}{x} + 9$;
$L'(x) = 2x + 4 - \frac{72}{x^3} - \frac{36}{x^2} = \frac{2}{x^3}(x^4 + 2x^3 - 18x - 36)$
$= \frac{2}{x^3}[(x^3 - 18)(x + 2)] = 0$ when $x = \sqrt[3]{18}$, since $x > 0$. Since $L'(x) < 0$ when $0 < x < \sqrt[3]{18}$ and $L'(x) > 0$ when $x > \sqrt[3]{18}$, L has an absolute minimum at $x = \sqrt[3]{18}$.



Thus the length of the ladder is $\sqrt{13 + \frac{54\sqrt[3]{18} + 108}{(\sqrt[3]{18})^2}} \doteq 7.0$ m.

Exercise 4.5

1. (a) $C(x) = 280000 + 12.5x + 0.07x^2$.
Average cost $= c(x) = \frac{C(x)}{x} = \frac{280000}{x} + 12.5 + 0.07x$, so the average cost for 1000 items is $c(1000) = 280 + 12.5 + 70 = \$362.50$/item.
Marginal cost $= C'(x) = 12.5 + 0.14x$, so the marginal cost for 1000 items is $C'(1000) = 12.5 + 140 = \$152.50$/item.
(b) $c'(x) = \frac{-280000 + .07x^2}{x^2} = 0$, when $x^2 = 4.0 \times 10^6$, so $x = 2000$. Also, $c'(x) < 0$ for $x < 2000$, $c'(x) > 0$ for $x > 2000$. Thus the average cost is the least at a production level of 2000 items.
(c) $c(2000) = \frac{280000}{2000} + 12.5 + 0.07(2000) = \$292.50$/item.

2. (a) $C(x) = 6400 + \frac{x}{10} + \frac{x^2}{1000}$. $c(x) = \frac{C(x)}{x} = \frac{6400}{x} + \frac{1}{10} + \frac{x}{1000}$, so the average cost for 3000 units is $c(3000) = \frac{6400}{3000} + \frac{1}{10} + \frac{3000}{1000} = \$5.23$/unit. $C'(x) = \frac{1}{10} + \frac{x}{500}$, so the marginal cost for 3000 units is $C'(3000) = \frac{1}{10} + \frac{3000}{500} = \$6.10$/unit.
(b) $c'(x) = -\frac{6400}{x^2} + \frac{1}{1000} = 0$ when $x^2 = 6400000$, so $x = 2530$. [OR: equate marginal cost and average cost.] Thus the average cost is the least at a production level of 2530 units.
(c) $c(2530) = \frac{6400}{2530} + \frac{1}{10} + \frac{2530}{1000} = \$5.16$/unit.

3. $C(x) = 48000 + 0.28x + 0.00001x^2$, $R(x) = 0.68x - 0.00001x^2$.
$C'(x) = 0.28 + 0.00002x$, $R'(x) = 0.68 - 0.00002x$. Maximum profit occurs when $C'(x) = R'(x)$, so $0.28 + 0.00002x = 0.68 - 0.00002x$, thus $x = \frac{0.4}{0.00004} = 10000$.
Therfore, maximum profits are obtained when 10000 cans of soup are sold.

4. $p(x) = \frac{30000 - x}{10000}$, $C(x) = 6000 + 0.8x$. $R(x) = \frac{30000x - x^2}{10000}$, so $R'(x) = 3 - \frac{x}{5000}$ and $C'(x) = 0.8$. Maximum profit occurs when $C'(x) = R'(x)$, so $3 - \frac{x}{5000} = 0.8$, thus $x = 11000$. Therefore, maximum profits occur when 11000 submarines are sold.

5. (a) The demand function is $p(x) = 10 - \frac{2}{6000}(x - 27000) = 19 - \frac{x}{3000}$.

(b) $R(x) = 19x - \frac{x^2}{3000}$; $R'(x) = 19 - \frac{x}{1500} = 0$ when $x = 19(1500) = 28500$.

This gives a maximum since R(x) is a parabola opening down.

$p(28500) = 19 - \frac{28500}{3000} = \$9.50$/ticket.

6. (a) The demand function is $p(x) = 50 + \frac{1}{100}(8000 - x) = 130 - \frac{x}{100}$.

(b) $R(x) = 130x - \frac{x^2}{100}$; $R'(x) = 130 - \frac{x}{50} = 0$ when $x = 130(50) = 6500$.

This gives a maximum since R(x) is a parabola opening down.

$p(6500) = 130 - \frac{6500}{100} = \$65.00$/camera.

7. $p(x) = 400 + 10(120 - x) = 1600 - 10x$. $R(x) = 1600x - 10x^2$;

$R'(x) = 1600 - 20x = 0$ when $x = \frac{1600}{20} = 80$.

So $p(80) = 1600 - 10(80) = \$800.00$/apartment.

8. $p(x) = 400 + 8(120 - x) = 1360 - 8x$. $R(x) = 1360x - 8x^2$;

$R'(x) = 1360 - 16x = 0$ when $x = \frac{1360}{16} = 85$.

Thus maximum revenue occurs when there are 85 passengers.

4.6 Review Exercise

1. (a) $f(x) = x - x^3$; $f'(x) = 1 - 3x^2 = 0$ when $x = \pm\frac{1}{\sqrt{3}}$, so $\pm\frac{1}{\sqrt{3}}$ are the critical numbers. So $f'(x) < 0$ when $x < -\frac{1}{\sqrt{3}}$ and when $x > \frac{1}{\sqrt{3}}$. $f'(x) > 0$ when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. So f increases on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and decreases on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$.
(b) $f(x) = x + x^3$; $f'(x) = 1 + 3x^2 = 0$ for no real value of x, so there are no critical numbers and since $f'(x) > 0$ for all x, f increases on $(-\infty, \infty)$.
(c) $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 3$; $f'(x) = 12x^3 - 24x^2 - 12x + 24$ $= 12(x^2 - 1)(x - 2) = 0$ when $x = \pm 1$, or 2, so the critical numbers are ± 1 and 2.

Interval	$x+1$	$x-1$	$x-2$	$f'(x)$	$f(x)$
$(-\infty, -1)$	$-$	$-$	$-$	$-$	decreases
$(-1,1)$	$+$	$-$	$-$	$+$	increases
$(1,2)$	$+$	$+$	$-$	$-$	decreases
$(2,\infty)$	$+$	$+$	$+$	$+$	increases

(d) $f(x) = \frac{2x+1}{2x-1}$; $f'(x) = \frac{2(2x-1) - 2(2x+1)}{(2x-1)^2} = \frac{-4}{(2x-1)^2} = 0$ for no value of x, so there are no critical numbers and since $f'(x) < 0$ for all x, f decreases on $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$.
(e) $f(x) = \frac{x^2}{x+1}$; $f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2} = 0$ when $x = -2$, or 0, so the critical numbers are -2 and 0. $f'(x) > 0$ when $x < -2$ and when $x > 0$. $f'(x) < 0$ when $-2 < x < 0$, but $x \neq -1$. Thus f increases on $(-\infty, -2)$ and $(0,\infty)$ and decreases on $(-2, -1)$ and $(-1,0)$.
(f) $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$; $f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = \frac{5(x-2)}{\sqrt[3]{x}} = 0$ when $x = 2$, so critical numbers are 0 and 2. $f'(x) > 0$ when $x < 0$ and when $x > 2$. $f'(x) < 0$ when $0 < x < 2$. Thus f increases on $(-\infty, 0)$ and $(2,\infty)$ and decreases on $(0,2)$.

2. (a) $f(x) = 4x^2 + 12x - 7$; $f'(x) = 8x + 12 = 0$ when $x = -\frac{3}{2}$. $f(-\frac{3}{2}) = -16$. Checking the endpoints, $f(-2) = -15$ and $f(1) = 9$.
So the absolute maximum is $f(1) = 9$ and the absolute minimum is $f(-\frac{3}{2}) = -16$.
(b) $f(x) = x^3 - 27x + 32$; $f'(x) = 3x^2 - 27 = 3(x+3)(x-3) = 0$ when $x = \pm 3$. $f(-3) = 86$, $f(3) = -22$. Checking the endpoints, $f(-4) = 76$ and $f(4) = -12$.
So the absolute maximum is $f(-3) = 86$ and the absolute minimum is $f(3) = -22$.
(c) $g(x) = 3x^5 - 50x^3 + 135x$; $g'(x) = 15x^4 - 150x^2 + 135 = 15(x^4 - 10x^2 + 9)$ $= 15(x^2 - 1)(x^2 - 9) = 0$ when $x = \pm 1$ or $x = \pm 3$. $g(-1) = -88$, $g(1) = 88$,

$g(-3) = 216$, $g(3) = -216$. Checking the endpoints, $g(-2) = 34$ and $g(4) = 412$. So the absolute maximum is $g(4) = 412$ and the absolute minimum is $g(3) = -216$.

(d) $g(x) = \frac{1+x}{1-x}$; $g'(x) = \frac{(1-x)+(1+x)}{(1-x)^2} = \frac{2}{(1-x)^2} = 0$ for no value of x.

Checking the endpoints, $g(2) = -3$ and $g(5) = -\frac{3}{2}$.

So the absolute maximum is $g(5) = -\frac{3}{2}$ and the absolute minimum is $g(2) = -3$.

3. (a) $f(x) = 7 + 72x + 3x^2 - 2x^3$;

$f'(x) = 72 + 6x - 6x^2 = -6(x^2 - x - 12) = -6(x+3)(x-4) = 0$ when $x = -3, 4$. $f'(x) < 0$ when $x < -3$ and when $x > 4$. $f'(x) > 0$ when $-3 < x < 4$. So $f(4) = 215$ is a local maximum and $f(-3) = -128$ is a local minimum.

(b) $f(x) = x^4 - 72x^2 + 10$; $f'(x) = 4x^3 - 144x = 4x(x^2 - 36) = 0$ when $x = 0, \pm 6$. $f'(x) < 0$ when $x < -6$ and when $0 < x < 6$. $f'(x) > 0$ when $-6 < x < 0$ and when $x > 6$. So $f(0) = 10$ is a local maximum and $f(\pm 6) = -1286$ are local minima.

(c) $f(x) = \sqrt{16 - x^2}$; $f'(x) = \frac{-x}{\sqrt{16 - x^2}} = 0$ when $x = 0$. $f'(x) > 0$ when $x < 0$ and $f'(x) < 0$ when $x > 0$. So $f(0) = 4$ is a local maximum.

(d) $f(x) = 12 - 2|x+3| = \begin{cases} 6 - 2x & x \geq -3 \\ 18 + 2x & x < -3 \end{cases}$; $f'(x) = \begin{cases} -2 & x \geq -3 \\ 2 & x < -3 \end{cases}$. $f'(x) < 0$ when $x > -3$ and $f'(x) > 0$ when $x < -3$, so $f(-3) = 12$ is a local maximum.

4. $f(x) = x^4 - 8x^3 + 22x^2 - 24x$

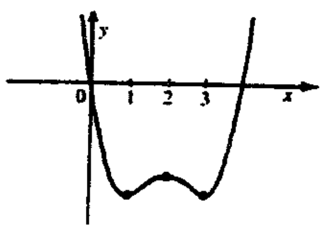
(a) $f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^2 + 11x - 6) = 4(x-1)(x-2)(x-3) = 0$ when $x = 1, 2$, or 3. So the critical numbers are 1, 2, and 3.

(b) $f'(x) = 4(x-1)(x-2)(x-3)$

Interval	$x-1$	$x-2$	$x-3$	$f'(x)$	$f(x)$
$(-\infty,1)$	−	−	−	−	decreases
$(1,2)$	+	−	−	+	increases
$(2,3)$	+	+	−	−	decreases
$(3,\infty)$	+	+	+	+	increases

(c) $f(2) = -8$ is a local maximum and $f(1) = -9$ and $f(3) = -9$ are local minima.

(d)



5. $f(x) = \frac{1}{x^2+x+1}$; $f'(x) = \frac{-(2x+1)}{(x^2+x+1)^2} = 0$ when $x = -\frac{1}{2}$. $f'(x) > 0$ when $x < -\frac{1}{2}$ and $f'(x) < 0$ when $x > -\frac{1}{2}$, so $f(-\frac{1}{2}) = \frac{4}{3}$ is an absolute maximum value.

6. Let x be the height and let y be the width of the printed material. $xy = 60$, so $y = \frac{60}{x}$. Area $= A(x) = (x+6)(y+10) = (x+6)(\frac{60}{x}+10) = 120 + 10x + \frac{360}{x}$;
$A'(x) = 10 - \frac{360}{x^2} = \frac{10(x^2-36)}{x^2} = 0$ when $x = \pm 6$, but -6 is not in the domain. Since $A'(x) < 0$ when $0 < x < 6$ and $A'(x) > 0$ when $x > 6$, the minimum area is $A(6) = (12)(20) = 240\ cm^2$.

7. Let x be the distance from the dimmer light to the dark spot, $0 < x < 40$. If I is illumination, S is the strength of the source, and d is distance then, $I = \frac{S}{d^2}$.
$I(x) = \frac{S}{x^2} + \frac{2S}{(40-x)^2}$; $I'(x) = -\frac{2S}{x^3} + \frac{2(2S)}{(40-x)^3} = 0$ when $2x^3 = (40-x)^3$, then $\sqrt[3]{2}x = 40 - x$, so $x = \frac{40}{1+\sqrt[3]{2}} \doteq 17.7$ m is the only critical point. $I'(x) < 0$ if $x < 17.7$ and $I'(x) > 0$ if $x > 17.7$, so I has a minimum at $x = 17.7$. Thus the darkest spot is approximately 17.7 m from the dimmer light.

8. Let 2x be the total distance along the x-axis and let 2y be the total distance along the y-axis of the rectangle. Area $= 4xy$, but from the equation of the ellipse $y = \frac{1}{2}\sqrt{4-x^2}$, thus $A(x) = 4x\left[\frac{1}{2}\sqrt{4-x^2}\right] = 2x\sqrt{4-x^2}$;
$A'(x) = 2\sqrt{4-x^2} - \frac{2x(2x)}{2\sqrt{4-x^2}} = \frac{2(4-x^2)-2x^2}{\sqrt{4-x^2}} = \frac{4(2-x^2)}{\sqrt{4-x^2}} = 0$ when $x = \pm\sqrt{2}$, but $-\sqrt{2}$ is not in the domain. $A'(x) > 0$ when $0 < x < \sqrt{2}$ and $A'(x) < 0$ when $x > \sqrt{2}$, so A has a maximum at $x = \sqrt{2}$ so $y = \frac{\sqrt{2}}{2}$. Thus the dimensions are $2\sqrt{2}$ by $\sqrt{2}$.

9. Let R be the length of road with R_1 the length of road from S to A and R_2 the length of road from S to B, x is the distance from D to the new station.
$R_1 = \sqrt{(x-6)^2+25}$ and $R_2 = \sqrt{x^2+49}$. $R(x) = \sqrt{(x-6)^2+25} + \sqrt{x^2+49}$;
$R'(x) = \frac{(x-6)}{\sqrt{(x-6)^2+25}} + \frac{x}{\sqrt{x^2+49}} = 0$ when $\frac{x^2}{x^2+49} = \frac{(6-x)^2}{(x-6)^2+25}$,
hence $x^2(x-6)^2 + 25x^2 = x^2(6-x)^2 + 49(6-x)^2$, then $25x^2 = 49(6-x)^2$, so

$5x = 7(6-x) = 42-7x$, thus $x = \frac{21}{6}$. $R(\frac{21}{6}) = \sqrt{\left(-\frac{15}{6}\right)^2+25}+\sqrt{\left(\frac{21}{6}\right)+49} \doteq 13.4$, and checking the endpoints $R(0) = \sqrt{36+25}+\sqrt{49} \doteq 14.8$, and $R(6) = \sqrt{25}+\sqrt{36+49} \doteq 14.2$. Therefore, a station should be located $\frac{21}{6}$ km from D.

10. $C(x) = 480\,000+2.4x+0.0008x^2$ and $p(x) = 4-0.001x$.

(a) $c(x) = \frac{480\,000}{x}+2.4+0.0008x$;

$c'(x) = -\frac{480\,000}{x^2}+0.0008 = \frac{0.0008x^2-480\,000}{x^2} = 0$ when $x = 10\,000\sqrt{6} \doteq 24\,495$.

Thus 24495 units will maximize the average cost.

(b) $C'(x) = 2.4+0.0016x$, $R(x) = 4x-0.001x^2$; $R'(x) = 4-0.002x$.

Profits are maximized when $C'(x) = R'(x)$, so $2.4+0.0016x = 4-0.002x$, then $x = \frac{1.6}{0.0036} \doteq 444$. So 444 units will maximize profits.

11. (a) $p(x) = 100-\frac{20}{30}(x-50) = \frac{400}{3}-\frac{2}{3}x = \frac{1}{3}(400-2x)$.

(b) $R(x) = \frac{1}{3}(400x-2x^2)$; $R'(x) = \frac{400}{3}-\frac{4x}{3} = \frac{4}{3}(100-x) = 0$ when $x = 100$.

Therefore, the revenue is maximized when 100 rooms are booked at $p(100) = \$66.67$.

4.7 Chapter 4 Test

1. $f(x) = \frac{x}{x^2+1}$; $f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = 0$ when $x = \pm 1$. $f'(x) > 0$ when $-1 < x < 1$. So f increases on $(-1,1)$.

2. $f(x) = x^3+2x^2+x-1$; $f'(x) = 3x^2+4x+1 = (3x+1)(x+1) = 0$ when $x = -\frac{1}{3}, -1$. $f(-1) = -1$, $f(-\frac{1}{3}) = -\frac{31}{27} \doteq -1.15$, and $f(1) = 3$.
So the absolute maximum is $f(1) = 3$ and the absolute minimum is $f(-\frac{1}{3}) = -\frac{31}{27}$.

3. $f(x) = x^4-8x^2+3$; $f'(x) = 4x^3-16x = 4x(x^2-4) = 4x(x-2)(x+2) = 0$ when $x = -2$, 0, or 2.

(a) The critical numbers are -2, 0, and 2.

(b)

Interval	x	$x-2$	$x+2$	f'	f
$(-\infty,-2)$	$-$	$-$	$-$	$-$	decreasing
$(-2,0)$	$-$	$-$	$+$	$+$	increasing
$(0,2)$	$+$	$-$	$+$	$-$	decreasing
$(2,\infty)$	$+$	$+$	$+$	$+$	increasing

(c) $f(0) = 3$ is a local maximum and $f(\pm 2) = -13$ are local minima.

4. Let x be the length of the sides of the base and let y be the height.
Volume $= V = x^2y$. $4x^2+2(4xy) = 4x^2+8xy = 1200$, so $y = \frac{1200-4x^2}{8x}$.
Thus $V(x) = \frac{x^2(1200-4x^2)}{8x} = 150x - \frac{x^3}{2}$; $V'(x) = 150 - \frac{3x^2}{2} = \frac{3(100-x^2)}{2} = 0$ when $x = \pm 10$. The length must be positive. $V'(x) > 0$ when $x < 10$ and $V'(x) < 0$ when $x > 10$, V has a maximum at $x = 10$ and $y = \frac{1200-400}{80} = 10$. Thus, the box is a cube with sides of 10 cm.

5. $C(x) = 16\,000 + 22.5x + 0.004x^2$. $c(x) = \frac{C(x)}{x} = \frac{16\,000}{x} + 22.5 + 0.004x$;
$c'(x) = -\frac{16\,000}{x^2} + 0.004 = 0$ when $x^2 = \frac{16\,000}{0.004}$, so $x = 2000$, since x must be positive. Therefore, the production level should be 2000 items.

6. Yield $= Y = (80+x)(400-4x) = -4x^2+80x+32\,000$;
$Y'(x) = -8x+80 = 0$ when $x = 10$. Therefore, 90 trees per hectare will give the largest crop.